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Representation of Harmonic Functions with Asymptomic Boundary Conditions

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ABSTRACT. The representation of a harmonic function outside a compact set in $\mathbb{R}^n$ is obtained, subject to a one-sided growth condition at infinity. By an inversion transformation, this result is used to characterize the behaviour of a harmonic function in the neighbourhood of an isolated singular point, and leads to a generalized version of Bôcher's theorem.

1. Introduction

In $\mathbb{R}^n$, $n \geq 2$, the fundamental singularity E_n of the Laplacian operator Δ at $x = 0$ is given by

$$E_n(x) = \begin{cases} \dfrac{1}{2\pi} \log |x| & \text{when } n = 2 \\ -\dfrac{1}{(n-2)\sigma_n} \dfrac{1}{|x|^{n-2}} & \text{when } n > 2, \end{cases}$$

σ_n being the area of the unit sphere in $\mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus \{0\}$. Thus, as a distribution in $\mathbb{R}^n$, E_n satisfies $\Delta E_n = \delta$, where δ is the Dirac measure in $\mathbb{R}^n$. As $|x| \to 0$ $E_n(x) \to -\infty$, but as $|x| \to \infty$ $E_n(x) \to 0$ if $n \geq 3$, while $E_2(x) \to \infty$. This discrepancy in the behaviour of E_n at infinity often requires a separate treatment for $n = 2$.

If K is a compact set in $\mathbb{R}^n$ and s is a harmonic function in $\mathbb{R}^n \setminus K$, we shall say that

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$s \in H_0(\mathbb{R}^n \setminus K)$ if s behaves like $E_n(x)$ as $|x| \to \infty$. More precisely, $s \in H_0(\mathbb{R}^n \setminus K)$ if there is a constant α such that

(i) when $n = 2$, $s(x) - \alpha \log |x| \to 0$ as $|x| \to \infty$.

(ii) when $n \geq 3$, $| s(x) | \leq | \alpha | / |x|^{n-2}$ as $|x| \to \infty$.

In Anandam and Al-Gwaiz (1993) it was proved in theorem 2.2 that if K is a compact set in $\mathbb{R}^n$, and Ω in an open set in K, then any harmonic function in $\Omega \backslash K$ can be represented uniquely as the sum of a function in $H_0(\mathbb{R}^n \backslash K)$ and a harmonic function in Ω. With $K = \{0\}$ and $\Omega = \mathbb{R}^n$, it provides the general expression for a harmonic function u in $\mathbb{R}^n \backslash \{0\}$ as

$$u(x) = \alpha E_n(x) + g(x) + h(x). \tag{1.1}$$

Here α is a constant which is determined by the flux of u at ∞, g is a harmonic function in $\mathbb{R}^n \backslash \{0\}$ which tends to 0 as $|x| \to \infty$, and h is a harmonic function in $\mathbb{R}^n$. The sum $\alpha E_n + g$ is, of course, the H_0 component of u. In $\mathbb{R}^2$, for example, we know from the theory of analytic functions that the general expression for a harmonic function in $\mathbb{R}^2 \backslash \{0\}$ is given by

$$u(r,\theta) = c \log r + \sum_{k=-\infty}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta)$$

where the negative powers of r constitute the function g, and the non-negative powers the function h.

But in this treatment, as in Anandam and Al-Gwaiz (1993), we have avoided using series expansions and have relied instead on the methods of potential theory, including the divergence theorem, the mean-value property for harmonic functions, and the existence of the Dirichlet solution. Our aim in this study is to characterize the behaviour of a harmonic function with a point singularity in $\mathbb{R}^n$. To simplify the notation the point is taken to be 0, but the results are obviously valid for any other point in $\mathbb{R}^n$. In general, the representation (1.1), based on Anandam and Al-Gwaiz (1993), is the best we can come up with. But if u is bounded on one side in a neighbourhood of 0, then the function g drops out. This is the content of a classical result due to Bôcher (1903).

In this article we prove a sharper result, where the one-sided boundedness at 0 is replaced by a one-sided growth condition. This is achieved by first obtaining the general expression for a harmonic function u outside a ball in $\mathbb{R}^n$, $n \geq 2$. As expressed by theorem 2.1, it is in fact a special case of theorem 2.2 in Anandam and Al-Gwaiz (1993) when $\Omega \backslash K$ is a spherical ring in $\mathbb{R}^n$, but it is all we need in this treatment. The proof is also much shorter than that of the general case. Theorem 2.2 gives the weakest growth condition which forces a harmonic function in $\mathbb{R}^n$ to be a constant; namely that the harmonic function should be bounded on one side by a function of order $o(|x|)$ as $|x| \to \infty$. The significance of this result is that this growth condition is imposed on the harmonic function rather than on its modulus, and that only potential theoretic methods are used to prove it.

Now theorems 2.1 and 2.2 lead to theorem 2.3, which gives sufficient (and necessary) conditions for the harmonic component of u to be a constant. Finally, the exterior of the ball is transformed by inversion to a punctured neighbourhood of 0, where the behaviour of u is characterized by theorem 3.1, a generalized version of Bôcher's theorem.

Viewed from a different perspective, Bôcher's theorem is shown to characterize a positive harmonic distribution whose Laplacian is supported in $\{0\}$. This point of view, which ultimately relies on the more powerful methods of distribution theory, provides a quick proof of Bôcher's theorem, and will probably be quite effective in tackling the more general problem of representing a polyharmonic function outside an isolated singularity, under appropriate growth behaviour near the singular point.

2. Harmonic Functions with Restricted Growth at ∞

If μ is a signed measure in $\mathbb{R}^n$ with compact support, the convolution product

$$E_n * \mu(x) = \int_{\mathbb{R}^n} E_n(\xi - x) d\mu(\xi)$$

is a well-defined function in $\mathbb{R}^n$, for all $n \geq 2$, which behaves like $\mu(\mathbb{R}^n) E_n(x)$ as $|x| \to \infty$, in the sense that $|E_n * \mu(x) - \mu(\mathbb{R}^n) E_n(x)| \to 0$ as $|x| \to \infty$.

On the other hand, if f is a continuous function on the circle $|x| = r$, for some positive number r, we shall use $D_r f$ to denote the Dirichlet solution in $|x| < r$ with boundary value f. In other words, $D_r f$ is the (unique) harmonic function in $|x| < r$ such that $D_r f(x) \to f(\xi)$ as $x \to \xi$, for any point ξ on the circle $|\xi| = r$.

Theorem 2.1

Let u be a harmonic function in $\{x \in \mathbb{R}^n : |x| > R\}$, where $n \geq 2$ and R is a positive number, and let $a > R$. Then $u(x) = E_n * \mu(x) + h(x)$ in $|x| > a$, where h is a harmonic function in $\mathbb{R}^n$ and μ is a signed measure with compact support in $\{x \in \mathbb{R}^n : |x| = a\} \cup \{0\}$.

Proof

Since $E_n(b) < E_n(a)$ whenever $R < b < a$, we can always choose $\alpha \geq 0$ so that

$$u(x) + \alpha E_n(b) \leq D_a u(x) + \alpha E_n(a) \qquad \text{on } |x| = b.$$

Noting that $D_a E_n(x) = E_n(a)$ in $|x| < a$ and that $u + \alpha E_n$ and $D_a(u + \alpha E_n)$ are equal on $|x| = a$, the maximum principle then implies that

$$u(x) + \alpha E_n(x) \leq D_a(u + \alpha E_n) \qquad \text{in } b < |x| < a.$$

By defining

$$s(x) = \begin{cases} u(x) + \alpha E_n(x) & \text{if } |x| \geq a \\ D_a(u + \alpha E_n)(x) & \text{if } |x| < a, \end{cases}$$

we see that s is subharmonic in $\mathbb{R}^n$ and harmonic outside the compact set $\{x \in \mathbb{R}^n : |x| = a\}$. Consequently there is a positive measure λ with support in the sphere $|x| = a$ such that $s(x) = E_n * \lambda(x) + h(x)$, where h is harmonic in $\mathbb{R}^n$.

Let $\mu = -\alpha\delta + \lambda$. Then supp $\mu \subset \{x \in \mathbb{R}^n : |x| = a\} \cup \{0\}$ and

$u(x) = s(x) - \alpha E_n(x) = E_n * \mu(x) + h(x)$ in $|x| > a$. □

Since $\Delta(E_n * \mu) = \delta * \mu = \mu$, it is clear that, in the terminology of Anandam and Al-Gwaiz (1993), $E_n * \mu$ is the H_0 component of u, which behaves like $\mu(\mathbb{R}^n)E_n(x)$ as $|x| \to \infty$.

We know that if a harmonic function in $\mathbb{R}^n$ is bounded below it can only be a constant. But in fact the same conclusion follows under weaker boundedness conditions, as the next result indicates.

Theorem 2.2

Let h be a harmonic function in $\mathbb{R}^n$ such that $h(x) \geq \phi(x)$ outside a compact set, where ϕ is a locally integrable function such that $\phi(x) = o(|x|)$ as $|x| \to \infty$, in the sense that $\phi(x) / |x| \to 0$ as $|x| \to \infty$. Then h is a constant.

Proof

Since $h(x) \geq \phi(x) \geq -|\phi(x)|$ outside a compact set, there is a positive number R such that $\upsilon(x) = -h(x) \leq |\phi(x)|$ for all $|x| \geq R$. But υ is harmonic in $\mathbb{R}^n$, so $\upsilon^+ = (1/2)(|\upsilon| + \upsilon)$ is a subharmonic function in $\mathbb{R}^n$ such that

$$\upsilon^+(x) \leq |\phi(x)| \qquad \text{in } |x| \geq R.$$

Let x_0 be any point in $\mathbb{R}^n$ and B_0 be any ball with centre x_0. Denoting the mean value of υ^+ on ∂B_0 by $M(\upsilon^+, \partial B_0)$, we therefore have

$$M(\upsilon^+, \partial B_0) \le M(|\phi|, \partial B_0).$$

Since $|\upsilon| = 2\upsilon^+ - \upsilon$, we also have

$$M(|\upsilon|, \partial B_0) = 2M(\upsilon^+, \partial B_0) - \upsilon(x_0) \le 2M(|\phi|, \partial B_0) - \upsilon(x_0). \tag{2.1}$$

Choose $B_0 = \{x \in \mathbb{R}^n : |x - x_0| < a\}$ with $a \ge |x_0| + R$, so that ∂B_0 lies in $|x| \ge R$. If $x = (x_1, \dots x_n)$ is the cartesian coordinate representation of $x \in \mathbb{R}^n$, then the partial derivatives $\partial_1 \upsilon = \partial\upsilon / \partial x_1, \dots, \partial_n \upsilon = \partial\upsilon/\partial x_n$ are all harmonic in $\mathbb{R}^n$, and the mean-value theorem gives

$$\partial_1 \upsilon(x_0) = \frac{1}{a^n \omega_n} \int_{B_0} \partial_1 \upsilon(x)\, dx, \tag{2.2}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Let $e_1, \dots, e_n$ be the unit vectors along $x_1, \dots, x_n$, respectively, and $\vec{\upsilon} = \upsilon e_1$. Using the divergence theorem, we can rewrite (2.2) as

$$\begin{aligned}\partial_1 \upsilon(x_0) &= \frac{1}{a^n \omega_n} \int_{B_0} \operatorname{div} \vec{\upsilon}\, dx \\ &= \frac{1}{a^n \omega_n} \int_{\partial B_0} \upsilon \cos\theta\, d\sigma,\end{aligned}$$

θ being the angle between e_1 and the exterior normal to ∂B_0 and $d\sigma$ the element of area on ∂B_0. Hence

$$\begin{aligned}|\partial_1 \upsilon(x_0)| &\le \frac{1}{a^n \omega_n} \int_{\partial B_0} |\upsilon|\, d\sigma \\ &= \frac{\sigma_n}{a \omega_n} M(|\upsilon|, \partial B_0) \\ &= \frac{n}{a} [2M(|\phi|, \partial B_0) - \upsilon(x_0)]\end{aligned}$$

in view of (2.1). Now the hypothesis on ϕ implies $(1/a) M(|\phi|, \partial B_0) \to 0$ as $a \to \infty$, and hence $\partial_1 \upsilon \equiv 0$. Similarly, $\partial_i \upsilon \equiv 0$ for all $i = 2, \dots, n$. Thus, υ, and consequently h, is constant. □

The fact that $h(x) = x_1$ is a non-constant-harmonic function in $\mathbb{R}^n$ clearly shows that the estimate of $o(|x|)$ cannot be improved upon.

Theorem 2.3

Let u be a harmonic function in $\{x\in \mathbb{R}^n : |x| > R\}$, where $n \geq 2$ and $R > 0$, and suppose that $u(x) \geq \phi(x)$ outside a compact set, with ϕ locally integrable and $\phi(x) = 0(|x|)$ as $|x| \to \infty$. If $a > R$, then there is a signed measure μ with support in $\{x\in \mathbb{R}^n : |x| = a\} \cup \{0\}$ such that

$$u(x) = E_n * \mu(x) + \text{a constant} \quad \text{in } |x| > a.$$

Proof

The function u satisfies the hypothesis of theorem 2.1, so we have $u(x) = E_n * \mu(x) + h(x)$ in $|x| > a$ for some harmonic function h in $\mathbb{R}^n$. Since $| E_n * \mu(x) | \leq \alpha | E_n(x) |$ outside a compact set for some constant $\alpha \geq 0$, it is clear that h satisfies the hypothesis of theorem 2.2 and is therefore a constant. □

Corollary

Let u(x) be harmonic in $|x|> R$, $x\in \mathbb{R}^n$ and $n \geq 3$. Then

i) Either u(x) is bounded on one side, in which case $\lim_{x\to\infty} u(x)$ exists,
ii) Or u(x) takes every real value infinitely often.

Proof

i) Suppose u(x) is lower bounded. Then, by the above theorem, for $R < a < b$, $u(x) = E_n * \mu(x) + \text{a constant}$ in $|x| > a$ and since $\lim_{x\to\infty} E_n * \mu(x) = 0$ if $n \geq 3$, we have the first alternative.

ii) Otherwise, $\overline{\lim}_{x\to\infty} u(x) = \infty$ and $\underline{\lim}_{x\to\infty} u(x) = -\infty$. Consequently, the image of $|x| > R$ under the continuous function u is a connected set which is necessarily the whole real line $\mathbb{R}$. Thus, if c is an arbitrary real number, there exists x_1 such that $u(x_1) = c$ and $|x_1| = R_1 > R$. The same argument shows that there is an x_2 such that $u(x_2) = c$ and $|x_2| = R_2 > R_1$.

Thus, for an infinite sequence of points (x_i), $|x_i| \to \infty$, $u(x_i) = c$. □

3. Harmonic Functions with Restricted Growth at 0

By mapping the region $|x| > 1$ onto $0 < |x| < 1$, theorem 2.3 can be used to characterize the local behaviour of a harmonic function in the neighbourhood of an isolated singularity, given the corresponding growth restriction at the singular point.

Theorem 3.1

Let u be a harmonic function in $\{x \in \mathbb{R}^n : 0 < |x| < 1\}$ and suppose $u(x) \geq \phi(x)$, where ϕ is locally integrable and $\phi(x) = o(1/|x|^{n-1})$ as $|x| \to 0$. Then there is a constant α and a harmonic function υ in $|x| < 1$ such that $u(x) = \alpha E_n(x) + \upsilon(x)$.

Proof

Denoting the unit open ball $\{x \in \mathbb{R}^n : |x| < 1\}$ by B_n, the transformation $x \to x / |x|^2$ maps $B_n \setminus \{0\}$ onto $B_n^* = \{x \in \mathbb{R}^n : |x| > 1\}$. If u^* is defined on B_n^* by the Kelvin transformation (see Helms 1969)

$$u^*(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right), \qquad x \in B_n^*, \tag{3.1}$$

it is a simple matter to verify that u^* is harmonic in B_n^*, where it satisfies

$$u^*(x) \geq \frac{1}{|x|^{n-2}} \phi\left(\frac{x}{|x|^2}\right) \quad \text{in } B_n^*. \tag{3.2}$$

When $n \geq 2$, the term $|x|^{2-n} \phi(x / |x|^2)$ is of order $o(|x|)$ as $|x| \to \infty$. Theorem 2.3 then implies that

$$u^*(x) = s(x) + \beta \quad \text{in } B_n^*$$

where $s = E_n * \mu$ and β is a constant.

Now the transformation (3.1) satisfies $u^{**} = u$, hence

$$u(x) = \frac{1}{|x|^{n-2}} u^* \left(\frac{x}{|x|^2}\right) = \frac{1}{|x|^{n-2}} \left[s\left(\frac{x}{|x|^2}\right) + \beta\right] \quad \text{in } B_n \setminus \{0\}.$$

As $|x| \to \infty$ we know that $s(x) - \gamma E_n(x) = o(1)$, where $\gamma = \mu(\mathbb{R}^n)$. Hence, as $|x| \to 0$, $s(x/|x|^2) - \gamma E_n(x / |x|^2) = o(|x|^{n-2})$. This implies that, as $|x| \to 0$,

$$u(x) = \frac{\gamma}{|x|^{n-2}} E_n\left(\frac{x}{|x|^2}\right) + \frac{\beta}{|x|^{n-2}} + o(1).$$

Since a bounded harmonic function in $B_n \setminus \{0\}$ extends to a harmonic function in B_n, we can therefore write

$$u(x) = \begin{cases} -\dfrac{\gamma}{2\pi} \log |x| + \upsilon(x) & \text{when } n = 2 \\ \\ \dfrac{\beta}{|x|^{n-2}} + \upsilon(x) & \text{when } n \geq 3, \end{cases}$$

for all $x \varepsilon B_n \setminus \{0\}$, where υ is harmonic in B_n. □

Remark:

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, the function $u(x) = x_1 / |x|^n$ is harmonic in $\mathbb{R}^n \setminus \{0\}$. This example shows that in the above theorem, the restriction that $\phi(x) = o(1/|x|^{n-1})$ is the best possible and we cannot even replace $o(1/|x|^{n-1})$ by $O(1/|x|^{n-1})$.

The special case when $\phi \equiv 0$ in theorem 3.1 yields the well known result.

Corollary (Bôcher's theorem)

If u is a non-negative harmonic function in $0 < |x| < 1$, then there is a constant $\alpha \leq 0$ and a harmonic function υ in $|x| < 1$ such that $u(x) = \alpha E_n(x) + \upsilon(x)$ in $0 < |x| < 1$.

The representation of u in Bôcher's theorem suggests an intimate connection with the solution of the differential equation $\Delta u = \delta$. This approach to the problem, based on the theory of distributions, provides a direct proof of Bôcher's theorem.

4. Another Proof of Bôcher's Theorem

Theorem 4.1

If u is a harmonic function in $B_n \setminus \{0\}$ where it satisfies the inequality $u(x) \geq \alpha E_n(x)$ for some constant α, then $u(x) = cE_n(x) + \upsilon(x)$ for some constant c and a harmonic function υ in B_n.

Proof

Let s(x) be the positive function $u(x) - \alpha E_n(x)$ in $0 < |x| < 1$. Then, with the definition $s(0) = \underline{\lim}_{x \to 0} s(x)$, s(x) extends as a superharmonic function in $|x| < 1$ (see Helms 1969, theorem 7.7 p. 130, or Brelot 1965, p. 39).

This means that $\Delta s \leq 0$ in the sense of distributions. But since a positive distribution is a Radon measure, and since $\Delta s = 0$ in $0 < |x| < 1$, $\Delta s = c\delta$ where δ is the Dirac measure and $c \leq 0$. This implies that $s(x) = cE_n(x) + \upsilon(x)$ in $|x| < 1$ where $\upsilon(x)$ is harmonic. □

Remark:

The condition in theorem 3.1 that $\phi(x) = o(1/|x|^{n-1})$ is less restrictive on u than that in theorem 4.1.

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تمثيل الدوال التوافقية تحت شروط حدية مقاربة

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وباستخدام تحويل انقلابي مناسب يمكن أن توظَّف هذه الصيَّغة لتشخيص سلوك الدالة التوافقية في جوار نقطة شاذة معزولة . وهذا بدوره يقود إلى صورة معمَّمة لنظرية بوكر .