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Integral Sum Graphs from Complete Graphs, Cycles and Wheels

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ABSTRACT. A graph G is an integral sum graph if there is a labelling θ of its vertices with distinct integers, so that for any two distinct vertices u and v, uv is an edge of G if and only if $\theta(u) + \theta(v) = \theta(w)$ for some vertex w. G is a sum graph if the labels are positive integers. For each graph G there is a minimum number $\sigma(G)$ such that $G \cup \sigma(G)K_1$ is a sum graph, and there is a minimum number $\zeta(G)$ such that $G \cup \zeta(G)K_1$ is an integral sum graph. In this paper, we prove a conjecture of Harary that $\zeta(K_n) = \sigma(K_n)$ for all K_n with $n \geq 4$. Also, we show that cycles C_n and wheels W_n are integral sum graphs for all $n \neq 4$.

All graphs in this paper are finite and have no loops or multiple edges. We follow in general the graph-theoretic notation and terminology of (Harary 1969, 1994).

A graph G is an integral sum graph if there is a labelling θ of its vertices with distinct integers, so that for any two distinct vertices u and v, uv is an edge of G if and only if $\theta(u) + \theta(v) = \theta(w)$ for some vertex w. The integral sum graph $G^+(S)$ of a finite subset $S \subset Z = \{..., -2, -1, 0, 1, 2, ...\}$ is the graph G(V,E) where V = S and uv $\in$ E if and only if $u + v \in S$. Thus, an integral sum graph is isomorphic to the integral sum graph of some $S \subset Z$. If Z is replaced by $N = \{1,2,3,...\}$, then we obtain sum graphs.

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Harary (1990, 1994) introduced these classes of graphs and noted that for each graph G there is a minimum number $\sigma(G)$ such that $G \cup \sigma(G)K_1$ is a sum graph, and there is a minimum number $\zeta(G)$ such that $G \cup \zeta(G)K_1$ is an integral sum graph. $\sigma(G)$ is the sum number of G, and $\zeta(G)$ is the integral sum number of G. Obviously $\zeta(G) \leq \sigma(G)$ for all graphs G. The sum numbers of complete graphs were derived by Bergstrand *et al.* (1989). The sum numbers of complete bipartite graphs were obtained by Hartsfield and Smyth (1992). Ellingham (1993) proved that $\sigma(T) = 1$ for all nontrivial trees. Harary (1994) noted that paths P_n and matchings mK_2 are integral sum graphs, and offered some problems. The purpose of this paper is to prove a conjecture of Harary (1994) that $\zeta(K_n) = \sigma(K_n)$ for all $n \geq 4$, and to show that cycles C_n and wheels W_n are integral sum graphs for all $n \neq 4$.

Complete graphs

Bergstrand *et al.* (1989) verified that $\sigma(K_3) = 2$, $K_3 \cup 2K_1 \cong G^+\{1,3,4,5,7\}$, and derived the following formula for $\sigma(K_n)$.

Theorem 1. (Bergstrand *et al.* 1989) For all positive integers $n \geq 4$, $\sigma(K_n) = 2n - 3$.

To realize $\sigma(K_n) = 2n - 3$, Bergstrand *et al.* (1989) labelled the vertices of K_n with $1 + 4(i - 1)$, $1 \leq i \leq n$, and labelled the isolated vertices with $2 + 4j$, $1 \leq j \leq 2n - 3$.

Subsequently, Harary (1994) conjectured that $\zeta(K_n) = \sigma(K_n)$ for all K_n with $n \geq 4$. The purpose of this section is to prove this conjecture.

Let $G = K_n \cup \zeta(K_n)K_1$, $n \geq 4$, and consider a labelling θ of the vertices of G which realizes G as an integral sum graph. Without loss of generality, we may assume that the vertices of K_n are labelled with the distinct integers $a_1, a_2, ..., a_n$ which satisfy $a_1 < a_2 < ... < a_n$. Then all the $\frac{n(n-1)}{2}$ sums $a_i + a_j$ for $i \neq j$, $1 \leq i, j \leq n$ occur as labels of the vertices of G.

Lemma 1. If the vertices of G are labelled as above, then
(i) the label of every vertex of G is distinct from zero,
(ii) $a_i \neq -a_j$ for all $1 \leq i, j \leq n$.

Proof.

(i) Since $a_1 < a_2 < ... < a_n$, then

$a_1 + a_2 < a_1 + a_3 < \dots < a_1 + a_n < a_2 + a_n < a_3 + a_n < \dots < a_{n-1} + a_n$.

Let $A = \{a_1 + a_2, a_1 + a_3, \dots, a_1 + a_n, a_2 + a_n, a_3 + a_n, \dots, a_{n-1} + a_n\}$. Then $|A| = 2n - 3$. Since $n \geq 4$, then $2n - 3 > n$ and consequently $\zeta(K_n) \geq 1$. Therefore, the label of every vertex of G is different from zero.

(ii) This follows directly from (i).

Let $S = \{s_1, s_2, \dots, s_m\} \subset Z$. For $r \in Z$, $r \neq 0$, we put $rS = \{rs_1, rs_2, \dots, rs_m\}$. It is easy to verify the following result.

Lemma 2. If $r \in Z$, $r \neq 0$, $S \subset Z$, then $G^+(rS) \cong G^+(S)$.

Now we assume that the labelling θ has the property that the vertices of K_n are labelled with the integers $c_1, c_2, \dots, c_p, b_1, b_2, \dots, b_q$ which satisfy $c_p < \dots < c_2 < c_1 < 0 < b_1 < b_2 < \dots < b_q$, $p \geq 1$, $q \geq 1$, $p + q = n$.

Lemma 3. If the vertices of G are labelled as above, then

(i) for $q \geq 2$, $c_1 + b_1$ and $c_1 + b_2$ are the labels of isolated vertices of G,

(ii) there exist no $i < j < k$ with $b_i + b_j = b_k$,

(iii) there exist no $i < j < k$ with $c_i + c_j = c_k$,

(iv) for $p \geq 2$, $c_1 + b_1$ and $c_2 + b_1$ are the labels of isolated vertices of G.

Proof.

(i) Since $c_1 < 0 < b_1$ then $c_1 < c_1 + b_1 < b_1$. Hence, $c_1 + b_1$ is the label of an isolated vertex of G. Now suppose that $c_1 + b_2 = b_1$. If $q = 2$ then $p \geq 2$, $(c_p + c_1) + b_2 = c_p + b_1$, which contradicts that $(c_p + c_1)$ is the label of an isolated vertex of G. If $q > 2$ then $c_1 + (b_2 + b_q) = b_1 + b_q$, which contradicts that $b_2 + b_q$ is the label of an isolated vertex of G. Thus $c_1 + b_2$ is the label of an isolated vertex of G.

(ii) It is clear that (ii) holds for $1 \leq q \leq 2$. For $q = 3$, if $b_1 + b_2 = b_3$ then $(c_1 + b_1) + b_2 = c_1 + b_3$, which contradicts that $c_1 + b_1$ is the label of an isolated vertex of G. For $q \geq 4$, the argument given below is similar to the proof of Lemma 1 in (Bergstrand *et al.* 1989) and fills a gap in the proof of Case 3 of that proof. We consider four cases.

Case 1. For $k < q$, if $b_i + b_j = b_k$ then $b_i + (b_j + b_q) = b_k + b_q$, which contradicts that $b_j + b_q$ is the label of an isolated vertex of G.

Case 2. $k = q$, $i > 1$ and there exists $m < i$ such that $b_m + b_i > b_q$.

If $b_i + b_j = b_q$ then $(b_m + b_i) + b_j = b_m + b_q$, which contradicts that $b_m + b_i$ is the label of an isolated vertex of G.

Case 3. $k = q$, $i > 1$ and for all $m < i$, $b_m + b_i \leq b_q$.

Let $b_i + b_j = b_q$. If $b_m + b_i$ is the label of some isolated vertex of G for some $m < i$ then, as in Case 2, we obtain a contradiction. Hence, for all $m < i$, $b_m + b_i$ is not the label of an isolated vertex of G. Let $s < i$. Then $b_s + b_i = b_r$ for some $r \leq q$. If $r = q$ then $b_s + b_i = b_q = b_i + b_j$. It implies that $s = j > i$ which is a contradiction. If $r < q$ then $b_s + (b_i + b_q) = b_r + b_q$, which contradicts that $b_i + b_q$ is the label of an isolated vertex of G.

Case 4. $k = q$, $i = 1$.

Since $q \geq 4$, then there is an index $t \notin \{1,j,q\}$ such that $1 < t < q$. If $b_1 + b_j = b_q$ then $b_t + b_j > b_1 + b_j = b_q$, and consequently $b_1 + (b_t + b_j) = b_t + b_q$, which contradicts that $b_t + b_j$ is the label of an isolated vertex of G.

(iii) Let S be the set of labels of the vertices of G. By Lemma 2, $G^+(-1S) \cong G^+(S)$. Moreover,

$$-b_q < \ldots < -b_2 < -b_1 < 0 < -c_1 < -c_2 < \ldots < -c_p.$$

Hence, by (ii), there exist no $i < j < k \leq p$ with $(-c_i) + (-c_j) = -c_k$. Therefore, there exist no $i < j < k \leq p$ with $c_i + c_j = c_k$.

(iv) Since $c_1 < 0 < b_1$ then $c_1 < c_1 + b_1 < b_1$. Hence, $c_1 + b_1$ is the label of an isolated vertex of G. If $c_2 + b_1 = c_1$ and $q \geq 2$ then $c_2 + (b_1 + b_2) = c_1 + b_2$, which contradicts that $b_1 + b_2$ is the label of an isolated vertex of G. If $q = 1$ then $p \geq 3$, $(c_p + c_2) + b_1 = c_p + c_1$, which contradicts that $c_p + c_2$ is the label of an isolated vertex of G.

Theorm 2. The integral sum number of complete graphs is given by

$$\zeta(K_n) = \begin{cases} 0 \text{ when } 1 \le n \le 3, \\ \sigma(K_n) = 2n - 3 \text{ when } n \ge 4. \end{cases}$$

Proof. It is clear that $K_1 \cong G^+\{1\}$, $K_2 \cong G^+\{0,1\}$, and $K_3 \cong G^+\{-1,0,1\}$. Thus $\zeta(K_n) = 0$ for $1 \le n \le 3$. Now, we use the notation of Lemma 3. If $q = 0$ then, by Theorem 1, the number of isolated vertices of G is greater than or equal to $2n - 3$. Let S be the set of labels of the vertices of G. If $p = 0$ then, by Lemma 2, $G^+(-1S) \cong G^+(S)$. Hence, by Theorem 1, the number of isolated vertices of G is greater than or equal to $2n - 3$. Now, suppose that $p \ge 1$ and $q \ge 1$. Let $C = \{c_1 + c_2, ..., c_1 + c_p, c_2 + cp, ..., c_{p-1} + c_p\}$ and $B = \{b_1 + b_2, ..., b_1 + b_q, b_2 + b_q, ..., b_{q-1} + b_q\}$. If $p = 1$ then $q \ge 3$, by Lemma 3, the set $B \cup \{c_1 + b_1, c_1 + b_2\}$ implies that the number of isolated vertices of G is greater than or equal to $2n - 3$. If $q = 1$ then, by Lemma 3, the set $C \cup \{c_1 + b_1, c_2 + b_1\}$ implies that the number of isolated vertices of G is greater than or equal to $2n - 3$. If $p \ge 2$ and $q \ge 2$ then, by Lemma 3, the set $C \cup B \cup \{c_2 + b_1, c_1 + b_1, c_1 + b_2\}$ implies that the number of isolated vertices of G is greater than or equal to $2n - 3$. Hence, $\zeta(K_n) \ge 2n - 3$. By Theorem 1, $\sigma(K_n) = 2n - 3$. It is obvious that $\zeta(K_n) \le \sigma(K_n)$. Therefore, $\zeta(K_n) = \sigma(K_n) = 2n - 3$.

Cycles and Wheels

Harary (1994) showed that all paths are integral sum graphs. $\{0\}$ realizes $\zeta(P_1) = 0$, $\{0,1\}$ realizes $\zeta(P_2) = 0$, and $\{0,1,2\}$ realizes $\zeta(P_3) = 0$. To realize $\zeta(P_n) = 0$ for $n \ge 4$, take the initial subsequence of order n of the sequence

$$(b_1, b_2, ...) = (1,2, -1,3, -4,7,...)$$

satisfying $b_n = b_{n-2} - b_{n-1}$ for $n \ge 3$, $b_1 = 1$ and $b_2 = 2$. Sequences which satisfy this recurrence relation and which are useful for realizing $\zeta(P_n) = 0$, may be obtained by requiring $b_1 + b_2$ to be a certain suitable term of the sequence. Besides (1, 2, – 1, 3, ...), here are two examples: (4, 1, 3, – 2, 5, –7, ...) may be used to label P_n for $n \ge 5$, and (9, 4, 5, – 1, 6, – 7, 13, – 20, ...) may be used to label P_n for $n \ge 7$. In what follows we will use (4, 1, 3, – 2, 5,– 7, ...). Rather than using the recurrence relation, we will view this sequence as derived from the Fibonacci sequence

$$(a_4, a_5, a_6, ...) = (2, 5, 7, ...)$$

satisfying $a_n = a_{n-2} + a_{n-1}$ for $n \ge 6$, $a_4 = 2$ and $a_5 = 5$, by setting $b_1 = 4$, $b_2 = 1$, $b_3 = 3$, and $b_n = (-1)^{n+1} a_n$ for $n \geq 4$. This view will be useful because certain properties of $(b_1, b_2, ...) = (4, 1, 3, -2, 5, ...)$ follow from the properties of Fibonacci sequences.

The following result will be used in the proof of the theorem about cycles.

Lemma 4. Let $P_n = v_1v_2 ... v_n$ be a path with n vertices $v_1, v_2, ..., v_n$. Define a labelling θ of the vertices of P_n as follows:

1) Choose two integers $\theta(v_1)$ and $\theta(v_2)$ such that $\theta(v_1)\,\theta(v_2) < 0$ and $|\theta(v_1)| < |\theta(v_2)|$,

2) For $3 \leq j \leq n$, define $\theta(v_j)$ by $\theta(v_j) = \theta(v_{j-2}) - \theta(v_{j-1})$.

If $\theta(u) + \theta(v) = \theta(w)$ then uv is an edge of P_n.

Proof. Notice that $\theta(v_1), \theta(v_2), ... \theta(v_n)$ is an alternating sequence, and $|\theta(v_1)| < |\theta(v_2)| < ... < |\theta(v_n)|$. Thus, for $2 \leq i \leq n$, $\theta(v_{i-1})\,\theta(v_i) < 0$ and consequently $|\theta(v_{i-1}) - \theta(v_i)| = |\theta(v_{i-1})| + |\theta(v_i)|$. Without loss of generality, we may assume that $|\theta(u)| < |\theta(v)|$.

First, we suppose that $w = v_j$ for some $3 \leq j \leq n$. We claim that $\theta(u)\theta(v) < 0$. To prove this claim, we suppose that $\theta(u)\theta(v) > 0$ and derive a contradiction. Since $\theta(u) + \theta(v) = \theta(w)$ and $\theta(u)\theta(v) > 0$, then $|\theta(u)| + |\theta(v)| = |\theta(w)|$. But $|\theta(u)| < |\theta(v)|$, so $|\theta(v)| > \dfrac{|\theta(w)|}{2}$. We have $\theta(w) = \theta(v_j) = \theta(v_{j-2}) - \theta(v_{j-1})$, and this gives $|\theta(w)| = |\theta(v_{j-2})| + |\theta(v_{j-1})| > 2|\theta(v_{j-2})|$. Hence, $|\theta(v_{j-2})| < |\theta(v)| < |\theta(v_j)|$. Therefore $v = v_{j-1}$, and $\theta(v)\theta(w) = \theta(v_{j-1})\theta(v_j) < 0$ which is a contradiction.

Since $\theta(u) + \theta(v) = \theta(w)$, $\theta(u)\theta(v) < 0$, and $|\theta(u)| < |\theta(v)|$, then $|\theta(v)| = |\theta(w) - \theta(u)| = |\theta(w)| + |\theta(u)|$. Clearly $v \neq v_1$ and $v \neq v_2$, so $v = v_j$ for some $3 \leq j \leq n$. Thus $\theta(v) = \theta(v_j) = \theta(v_{j-2}) - \theta(v_{j-1})$, and this gives $|\theta(v)| = |\theta(v_{j-2})| + |\theta(v_{j-1})| > 2|\theta(v_{j-2})|$. If $|\theta(u)| < |\theta(w)|$ then $|\theta(v)| < 2|\theta(w)|$. Hence, $|\theta(v_{j-2})| < |\theta(w)| < |\theta(v_j)|$, and consequently $w = v_{j-1}$ which contradicts the alternating nature of $\theta(v_1), \theta(v_2), ..., \theta(v_n)$. Therefore $|\theta(w)| < |\theta(u)|$. Repeating the previous argument we obtain $u = v_{j-1}$, and consequently uv is an edge of P_n.

Second, we suppose that $w = v_1$. Since $|\theta(v_1)| < |\theta(v_2)| < \ldots$ then $\theta(u)\theta(v) < 0$; hence $|\theta(v)| = |\theta(v_1) - \theta(u)| = |\theta(v_1)| + |\theta(u)|$. Obviously $|\theta(v_1)| < |\theta(u)| < |\theta(v)|$, so $v = v_j$ for some $3 \le j \le n$. Repeating the previous argument we obtain $u = v_{j-1}$. Thus $\theta(v_{j-1}) + \theta(v_j) = \theta(v_1)$, so $j = 3$. Therefore $uv = v_{j-1}\, v_j$ is an edge of P_n.

Third, we suppose that $w = v_2$. Then $\theta(u) + \theta(v) = \theta(v_2)$. If $u = v_1$, then $\theta(v) = \theta(v_2) - \theta(v_1) = -\theta(v_3)$, which contradicts that $|\theta(v_1)| < |\theta(v_2)| < \ldots$. Thus $u = v_k$, $v = v_j$ for some $3 \le k < j \le n$. If $\theta(u)\theta(v) > 0$ then $|\theta(u)| + |\theta(v)| = |\theta(v_2)|$, which contradicts that $|\theta(v_1)| < |\theta(v_2)| < \ldots$. Thus $\theta(u)\theta(v) < 0$, and consequently we obtain $|\theta(v_j)| = |\theta(v_2) - \theta(u)| = |\theta(v_2)| + |\theta(u)|$. Then, by applying the previous argument, we have $u = v_{j-1}$. Thus $\theta(v_j) = \theta(v_2) - \theta(v_{j-1})$; since $|\theta(v_1)| < |\theta(v_2)| < \ldots$, then $j = 4$ and consequently $uv = v_3v_4$ is an edge of P_n.

Theorem 3. The integral sum number of cycles is given by

$$\zeta(C_n) = \begin{cases} 3 & \text{when } n = 4 \\ 0 & \text{when } n \neq 4. \end{cases}$$

Proof. Harary (1994) remarked that $\zeta(C_4) = \sigma(C_4) = 3$ and noted that $\{1, 5, 9, 13, 6, 14, 22\}$ realizes $\zeta(C_4) = 3$. For completeness, we give a proof of this fact. Let $G = C_4 \cup \zeta(C_4)\, K_1 \cong G^+(S)$. It is clear that $0 \notin S$, and consequently the sum of any edge of C_4 is different from zero (the sum of an edge uv is $\theta(u) + \theta(v)$ where $\theta(u)$ and $\theta(v)$ are the labels of u and v respectively). Assume that the vertices of C_4 are labelled as in Figure 1 (i). If $a + b = d + c$ and $a + d = b + c$, then $a = c$ gives a contradiction.

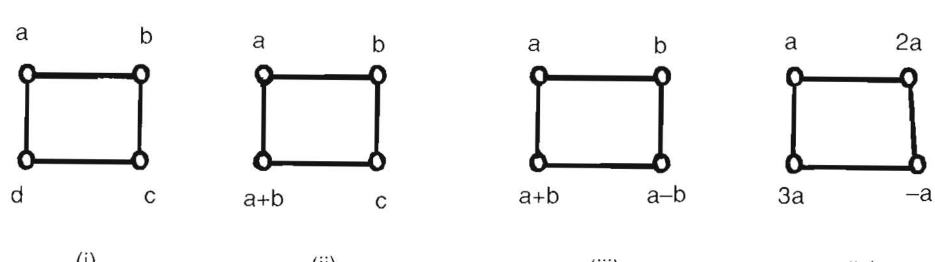

Figure 1

Thus, either $a + b \neq d + c$ or $a + d \neq b + c$. We claim that the sum of every edge of C_4 does not belong to $\{a, b, c, d\}$. For the sake of a contradiction, we may assume without loss of generality that the vertices of C_4 are labelled as in Figure 1 (ii). Clearly, $a + b + c \in S$ and $b + c \notin \{b, c, a + b\}$. If $b + c \neq a$ then $a + (b + c) = a + b + c$, which contradicts that $b + c$ is the label of an isolated vertex of G. If $b + c = a$ then $c = a - b$, and we obtain Figure 1 (iii). Obviously $2a \notin \{a, a + b, a - b\}$. If $2a \neq b$ then $b + (2a) = 2a + b$, which contradicts that $2a$ is the label of an isolated vertex of G. If $2a = b$ then we obtain Figure 1 (iv), and $-a + (4a) = 3a$, which contradicts that $4a$ is the label of an isolated vertex of G. This completes the proof of the claim. Hence, $\zeta(C_4) \geq 3$. It is obvious that $\zeta(C_4) \leq \sigma(C_4)$ and $C_4 \cup 3K_1 \cong G^+\{1, 5, 9, 13, 6, 14, 22\}$. Therefore $\zeta(C_4) = \sigma(C_4) = 3$.

Now, we consider some special cases. We have

$$C_3 \cong G^+\{-1, 0, 1\},$$
$$C_5 \cong G^+\{1, 2, -1, 3, -2\},$$
$$C_6 \cong G^+\{-6, 5, -4, -1, -5, 1\},$$
$$C_7 \cong G^+\{4, 3, 1, 2, -5, 7, -3\},$$
$$C_9 \cong G^+\{-1, -3, -4, 1, -15, 8, -7, 15, -14\},$$
$$C_{11} \cong G^+\{-1, 4, 3, 1, -23, 15, -8, 7, -6, 21, -22\}.$$

In what follows we assume that n is a positive integer such that $n \notin \{3, 4, 5, 6, 7, 9, 11\}$. To realize $\zeta(C_n) = 0$ we use the sequence

$$(b_1, b_2, \ldots) = (4, 1, 3, -2, 5, -7, \ldots)$$

satisfying $b_n = b_{n-2} - b_{n-1}$ with $b_1 = 4$ and $b_2 = 1$. We put $d_{n-1} = b_1 + b_2 - b_{n-2} = 5 - b_{n-2}$ and $d_n = b_{n-2} - b_1 = b_{n-2} - 4$. We claim that

$$C_n \cong G^+\{b_1, b_2, \ldots, b_{n-2}, d_{n-1}, d_n\}.$$

The labelling is illustrated below (see Fig. 2).

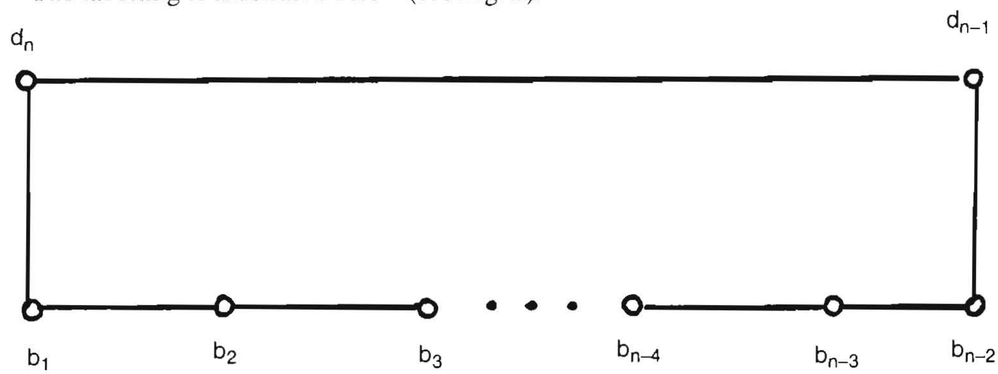

Figure 2

The proof of this claim is easy but cumbersome. Before giving it, we demonstrate the algorithm on the following two examples (see Fig. 3).

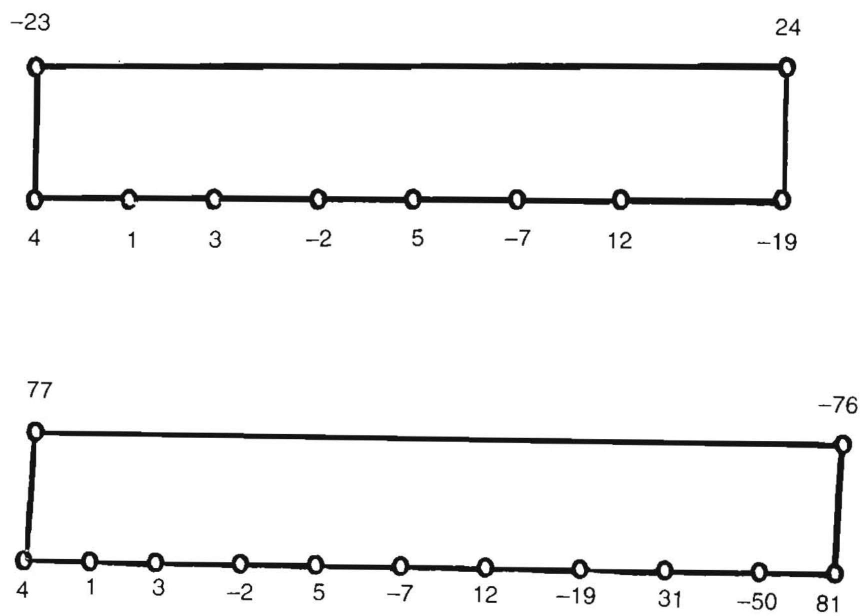

Figure 3

It is easy to verify the claim for $n \in \{8, 10\}$. So we assume that $n \geq 12$. If n is even then $b_{n-2} < 0$. Thus $|d_{n-1}| = |5 - b_{n-2}| = 5 + |b_{n-2}|$ and $|d_n| = |b_{n-2} - 4| = |b_{n-2}| + 4$. Hence, for n even, we have

$$0 < |b_2| < |b_4| < |b_3| < |b_1| < |b_5| < |b_6| < \ldots < |b_{n-2}| < |d_n| < |d_{n-1}| < |b_{n-1}| \ldots \quad (1)$$

If n is odd then $b_{n-2} > 0$. Thus $|d_{n-1}| = |b_{n-2}| - 5$ and $|d_n| = |b_{n-2}| - 4$. Obviously $|b_{n-2}| - |b_{n-3}| > 5$; so , for n odd, we have

$$0 < |b_2| < |b_4| < |b_3| < |b_1| < |b_5| < |b_6| < \ldots < |b_{n-3}| < |d_{n-1}| < |d_n| < |b_{n-2}| \ldots \quad (2)$$

Let $x + y = z$ where $x, y, z \in S = \{b_1, b_2, ..., b_{n-2}, d_{n-1}, d_n\}$, and without loss of generality assume that $|x| < |y|$.

If $z = 1$ then $x + y = 1$, which implies that $x < 0$ and $y > 0$. Hence $y = |y| = |x| + 1$. Thus, by (1) and (2), either $x = -2$, $y = 3$ or $x = d_{n-1}$, $y = d_n$ for n even and $x = d_n$, $y = d_{n-1}$ for n odd.

If $z = -2$ then $x + y = -2$, which implies that $x > 0$ and $y < 0$. Thus $|y| = |x| + 2$. Hence, by (1) and (2), $x = 5$, $y = -7$.

Similarly, we use (1) and (2) implicitly in the discussion of the following cases.

If $z = 3$ then $x + y = 3$. If $xy > 0$ then $x = 1, y = 2 \notin S$. If $xy < 0$ then $x < 0$ and $y = |y| = |x| + 3$. Thus $x = -2$, $y = 5$.

If $z = 4$ then $x + y = 4$. If $xy > 0$ then $x = 1$, $y = 3$. If $xy < 0$ then $y = |y| = |x| + 4$ which has no solution in $S = \{b_1, b_2, ..., b_{n-2}, d_{n-1}, d_n\}$.

If $z = 5$ then $x + y = 5$. If $xy > 0$ then $x = 1$, $y = 4$ or $x = 2 \notin S$, $y = 3$. If $xy < 0$ then $y = |y| = |x| + 5$. Hence either $x = -7$, $y = 12$ or $x = b_{n-2}$, $y = d_{n-1}$ for n even and $x = d_{n-1}$, $y = b_{n-2}$ for n odd.

If $z = -7$ then $x + y = -7$ which has no solution in S when $x < 0$ and $y < 0$. If $xy < 0$ then $x > 0$, $y < 0$ and $|y| = |x| + 7$. Thus, $x = 12$, $y = -19$.

Now, let $z \in \{b_7, b_8, ..., b_{n-2}\}$. Clearly, $12 \leq |z| \leq |b_{n-2}|$. Thus $x + y = z$ has no solution in $\{b_1, b_2, ..., b_6\}$. Also $d_{n-1} + d_n = 1 \neq z$, and if $x,y \in \{b_7, b_8, ..., b_{n-2}\}$ then it follows from Lemma 4 that xy is an edge of $G^+\{b_7, b_8, ..., b_{n-2}\}$. By definition, $b_1 + d_n = b_{n-2}$, $b_{n-2} + d_{n-1} = b_5$, and $d_{n-1} + d_n = b_2$. Let $x = b_i$, $y = b_j$, $z = b_k$ for some $1 \le i \le 6$, $7 \le j, k \le n-2$. Thus $b_i + b_j = b_k$, which implies that $b_i + (-1)^{j+1} a_j = (-1)^{k+1} a_k$. We observe that $|b_i| \le 7$, $a_j \ge 12$, $a_k \ge 12$, $|a_k - a_j| \ge 7$, and consider four cases according to parity. If both j and k are odd then $b_i = a_k - a_j$ which occurs only when $a_k = b_7$, $a_j = -b_8$, $b_i = b_6$, and this solution is rejected because $a_j = -b_8$ implies that $j = 8$ which is even. If both j and k are even then $b_i = a_j - a_k$ which occurs only when $a_j = b_7$, $a_k = -b_8$, $b_i = b_6$, and this solution is rejected because $a_j = b_7$ implies that $j = 7$ which is odd. If j is odd and k is even then $b_i = -(a_j + a_k)$ which cannot occur because $a_j + a_k > |b_i|$. If j is even and k is odd then $b_i = a_j + a_k$ which cannot occur. Clearly $b_{n-2} + d_i \ne b_j$ for all $7 \le j \le n-2$, $n-1 \le i \le n$. Let $y = d_i$, $x = b_j$, $z = b_k$ for some $n-1 \le i \le n$, $7 \le j \le n-3$, $7 \le k \le n-2$. Thus $b_j + d_i = b_k$, which implies that $d_i = (-1)^{k+1}a_k - (-1)^{j+1}a_j$. Thus

$$|(-1)^{k+1}a_k - (-1)^{j+1} a_j| = \begin{cases} a_{n-2} + 5 & \text{when } i = n-1 \text{ and n is even,} \\ a_{n-2} - 5 & \text{when } i = n-1 \text{ and n is odd,} \\ a_{n-2} + 4 & \text{when } i = n \text{ and n is even,} \\ a_{n-2} - 4 & \text{when } i = n \text{ and n is odd,} \end{cases}$$

which is impossible.

It remains to consider $x + y = d_i$, $n-1 \le i \le n$. Let $x = b_j$, $y = b_k$, $1 \le j < k \le n-2$. Then, for $4 \le j < k \le n-2$, we have

$$|(-1)^{j+1}a_j + (-1)^{k+1} a_k| = \begin{cases} a_{n-2} + 5 & \text{when } i = n-1 \text{ and n is even,} \\ a_{n-2} - 5 & \text{when } i = n-1 \text{ and n is odd,} \\ a_{n-2} + 4 & \text{when } i = n \text{ and n is even,} \\ a_{n-2} - 4 & \text{when } i = n \text{ and n is odd,} \end{cases}$$

which is impossible. For $1 \le j \le 4$ and $1 \le j < k \le n-2$, $b_j + b_k = d_i$ is impossible. Finally, $x + d_j = d_i$ is impossible. This completes the proof.

Recall that the wheel W_n is defined by $W_n = K_1 + C_{n-1}$ for $n \ge 4$.

Theorem 4. The integral sum number of wheels is given by

$$\zeta(W_n) = \begin{cases} 5 & \text{when } n = 4, \\ 0 & \text{when } n \ne 4. \end{cases}$$

Proof. First, we consider some special cases. By Theorem 2, since $W_4 \cong K_4$, then $\zeta(W_4) = \zeta(K_4) = 5$ and $W_4 \cup 5K_1 \cong G^+ \{1, 5, 13, 9, 6, 10, 14, 18, 22\}$. It is easy to verify that

$$\begin{aligned}
W_5 &\cong G^+ \{0, -1, 1, -2, 2\}, \\
W_6 &\cong G^+ \{0, -1, 1, 3, -3, 4\}, \\
W_7 &\cong G^+ \{0, 1, 3, -2, 5, -4, 4\}, \\
W_8 &\cong G^+ \{0, 1, 6, -5, 4, -3, 7, -1\}, \\
W_{10} &\cong G^+ \{0, 1, 6, -5, 4, -9, 16, -16, 7, -1\}, \\
W_{12} &\cong G^+ \{0, 1, 6, -5, 4, -9, 15, -27, 27, -12, 7, -1\}.
\end{aligned}$$

Second, for $n + 1 \notin \{4, 5, 6, 7, 8, 10, 12\}$, we consider the set $S = \{b_1, b_2, ..., b_{n-2}, d_{n-1}, d_n\}$ which was defined in the proof of Theorem 3. We claim that if x and y belong to S, then $x + y \neq 0$. Indeed, as in the proof of Theorem 3, we have

$$0 < |b_2| < |b_4| < |b_3| < |b_1| < |b_5| < |b_6| < ... < |b_{n-2}| < |d_n| < |d_{n-1}| < |b_{n-1}|$$

whenever n is even, and

$$0 < |b_2| < |b_4| < |b_3| < |b_1| < |b_5| < |b_6| < ... < |b_{n-3}| < |d_{n-1}| < |d_n| < |b_{n-2}|$$

whenever n is odd. Thus, if $x, y \in S$ and $x \neq y$, then $|x| \neq |y|$ and consequently $x + y \neq 0$. Hence,

$$W_{n+1} \cong G^+ \{0, b_1, b_2, ..., b_{n-2}, d_{n-1}, d_n\}.$$

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رسومات المجموع الصحيح
المشتقة من الرسومات التامة ، الدورات ، والدواليب

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يقال عن رسم G إنه رسم مجموع صحيح اذا كانت توجد عنونة θ لرؤوسه بأعداد صحيحة مختلفة بحيث يتحقق الشرط التالي : لأي رأسين مختلفين u , v فإن uv ضلع في G اذا وفقط اذا كان يوجد رأس w بحيث $\theta(u) + \theta(v) = \theta(w)$. G رسم مجموع اذا كانت العناوين اعدادا صحيحة موجبة . لكل رسم G يوجد عدد أصغر $\sigma(G)$ بحيث $G \cup \sigma(G)K_1$ رسم مجموع ، ويوجد عدد أصغر $\zeta(G)$ بحيث $G \cup \zeta(G)K_1$ رسم مجموع صحيح .

في هذا البحث ، نثبت صواب مخمنة لهراري (Harary 1994) بأن $\zeta(K_n)=\sigma(K_n)$ لكل $n\geq4$ ، كما نثبت أن الدورات C_n والدواليب W_n رسومات مجموع صحيح لكل $n\neq4$.