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Coset Diagrams and Relations for PSL (2,Z)

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ABSTRACT. A diagramatic argument, called coset diagrams for the modular group PSL(2,Z), is used to prove the results stated in this paper.

Let G denote the subgroup of the modular group PSL(2,Z), generated by the linear-fractional transformations x and y where x and y are respectively defined as $z \rightarrow -1/z$ and $z \rightarrow (z-1)/z$.

A diagram with n vertices depicts a (transitive) permutation representation of the modular group: fixed points of x and y are defined by heavy dots, and 3-cycles of y by triangles whose vertices are permuted anti-clockwise by y; and any two vertices which are interchanged by x are joined by an edge.

In this paper we have shown that the coset diagram for the action of G on the rational projective line is connected, and transitive. Using these coset diagrams we have shown that the group PSL(2,Z) is generated by the linear-fractional transformations x and y and that $x^2 = y^3 = 1$ are defining relations for PSL(2,Z).

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In this paper, we have shown that the coset diagram for the natural action of G, on the rational projective line, is connected and the action is transitive. A new proof is given to show that PSL(2,Z) is generated by the linear-fractional transformations x and y and that $x^2 = y^3 = 1$ are defining relations for PSL(2,Z).

If t is the transformation $z \rightarrow 1/z$ so that t belongs not to the modular group PSL(2,Z) but to PGL(2,Z) then x,y,t satisfy: $x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1$ (i)

Once it is shown that G has $x^2 = y^3 = 1$ as defining relations and that G is the whole of PSL (2,Z), it is clear that relations (i) are defining relations for $G^\star = \langle x,y,t \rangle$ and that $G^\star$ is the whole of PGL(2,Z).

This theory of coset diagrams has also proved useful in determining generators of PSL(2,p) or PGL(2,p), where p is a prime number. Coset diagrams for the action of the modular group★, on projective lines over F_p, gives some interesting information.

Before we come to our main results we shall make the following remarks:

(i) If $k \neq 1,0,\infty$ then of the vertices k, ky, ky^2 of a triangle, in a coset diagram for the action of PGL(2,Z) on any subset of the real projective line, one vertex is negative and two are positive.

(ii) Let $k = \pm\, a/b$ where a,b are positive integers with no common factor. For $k \neq 0,\infty$ we define $||k|| = \max(a,b)$. Clearly, $||k|| = ||kx||$ and if k is negative, then $||k||$ is less than $||ky||$ and $||ky^2||$.

(iii) We shall use arrow head on an edge to indicate its direction from negative to positive vertex.

Theorem 1

The action of PGL(2,Z) *on the rational projective line is connected.*

Proof

To prove this we need only to show that for any rational number k there is a path joining k to ∞.

Since one of k,kx is negative, therefore without any loss of generality, we can assume that k is negative.

Let $k = k_0$ be a negative rational number. Then k_0x is positive and if $k_0x \neq 1$ then by remark (i) there is just one negative number (vertex), say k_1, in the triangle containing k_0x, which cannot be k_0x. That is, we have a fragment of the coset diagram of one of the forms:

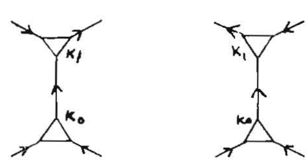

Fig. 1

By remark (ii) we note that $\|k_0\| > \|k_1\|$. If we now consider k_1 then k_1xy and k_1xy^2 will be the vertices of the triangle containing k_1x as its third vertex. Since k_1x is positive, therefore as in the case of k_0, of k_1xy and k_1xy^2 we get just one negative vertex, say k_2 such that $\|k_1\| > \|k_2\|$. If we continue like this and follow the arrows from $k = k_0$ in Fig. 2, we get a sequence of negative rational numbers $k_0, k_1, k_2, \ldots$ such that $\|k_0\| > \|k_1\| > \|k_2\| \ldots$

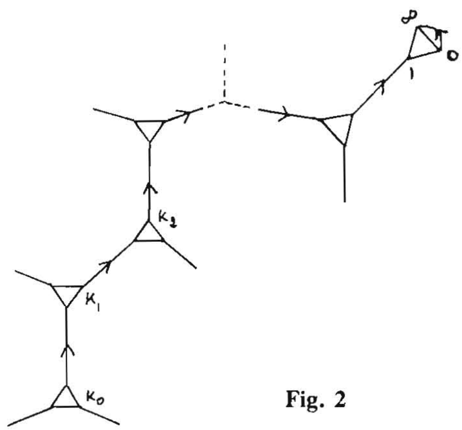

Fig. 2

The decreasing sequence of positive integers must terminate, and it can terminate only because ultimately we reach a triangle with the vertices 1,0 and ∞.

Fig. 3

A sequence of negative rational numbers $k_0, k_1, k_2, \ldots$ such that $\|k_0\| > \|k_1\| > \|k_2\| \ldots$ shows that there is a path joining $k = k_0$ to ∞. This implies that every rational number occurs in the diagram and that the diagram for the action of G on the rational projective line is connected.

Corollary 2

The action of G *on the rational projective line is transitive.*

Proof

We shall prove transitivity of the action, by showing that, if there is a path from a rational number p to a rational number q then there exists some g in G such that $pg = q$.

If $pg = q$ then $pxxg = q$ and since one of p,px is negative, we can assume without any loss of generality that p is negative.

Let $p = k_0$ and $q = k_i$ for some i. Then from the coset diagram (Fig: 2) we note that each k_{j+1} is either $k_j xy$ or $k_j xy^2$. This implies that $q = p\ xy^{\epsilon_1}\ xy^{\epsilon_2} \dots xy^{\epsilon_i}$ where each $\epsilon_j = 1$ or 2. If $xy^{\epsilon_1}\ xy^{\epsilon_2} \dots xy^{\epsilon_i} = g$ then $q = pg$, where q is in G. So the action of G on the rational projective line is transitive.

Theorem 3

The group PSL(2,Z) *is generated by the linear-fractional transformations* x *and* y.

Proof

Let h in PSL (2,Z) be such that $k = \infty h$ for a rational number k. By corollary 2, since the action of G on the rational projective line is transitive, therefore $kg = \infty$ for some g in G. So $\infty = kg = (\infty h)g$. Hence ∞ is a fixed point of hg. This means that hg is a linear-fractional transformation $z \to (az + b)/(CZ + d)$ with $c = 0$. Since $ad - bc = 1$, therefore $a = d = \pm 1$ which then implies that hg is $z \to z \pm b$ and $xy : z \to z + 1$ further implies that $hg = (xy)^{\pm b}$. This shows that hg and hence h is in G, proving that x and y are generators of the group G.

Theorem 4

Relations $x^2 = y^3 = 1$ *are defining relations for* PSL(2,Z).

Proof

Suppose $x^2 = y^3 = 1$ are not defining relations of PSL(2,Z). Then there is a relation of the form $xy^{\epsilon}\ xy^{\epsilon_2} \dots xy^{\epsilon_i} = 1$ where $m \geq 1$, $\epsilon_i = \pm 1$ and $i = 1,2,\dots,m$. We know that neither x nor y can be 1.

The coset diagram (Fig.2) depicts that it does not contain any closed circuit, apart from the circuit in the triangle containing ∞. For if it contains the closed circuit (Fig. 4). and $k_0, k_1, k_2, \dots, k_m$ are the vertices of the triangles in the diagram such that $k_0 < 0$, then this leads to the contradiction $||k_0|| > ||k_1|| > \dots ||k_m|| > ||k_0||$. So the coset diagram (Fig. 2) does not contain any other closed circuit, apart from the circuit in the triangle containing ∞ as its vertex.

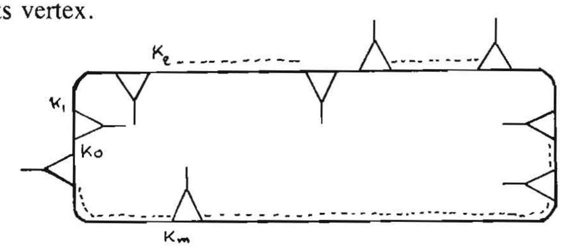

Fig. 4

This shows that there are points in the diagram whose distance from the point ∞ is arbitrary large. Choose k with $k<0$, so that the distance fom the point k to the point ∞ is greater than m. Define $k_i = k\ xy^{\epsilon_1}\ xy^{\epsilon_2} \dots xy^{\epsilon_i}$ where $i = 0,1,2,\dots,m$. Then $||k_0|| > ||k_1|| > ||k_2|| \dots > ||k_m||$ and in particular $k_m \neq k_0$. Thus $xy^{\epsilon_1}\ xy^{\epsilon_2} \dots xy^{\epsilon_m} \neq 1$ and so $x^2 = y^3 = 1$ are defining relations for the modular group PSL(2,Z).

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علاقات للزمرة PSL(2,Z)

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استعملت طريقة البرهان الرسمي التي تسمى الرسوم للمجموعات المشاركة للزمرة النموذج PSL (2, Z) ، للبرهنة على النتائج التي تم الحصول عليها في هذا البحث.

لتكن G زمرة جزئية من الزمرة النموذج PSL (2,Z) مولدة من قبل التحويلات الكسرية الخطية y, x معرفة على التوالي
$z \to (z-1)/z, z \to -1/z$

في هذا البحث برهنا على أن مجموعة الرسم للمجموعات المشاركة لتأثير G على الخط الكسرى الإسقاطي تكون متراصة ومتعدية. باستكمال مجموعة الرسوم للمجموعات المشاركة برهنا على أن الزمرة PSL (2,Z) مولدة من قبل التحويلات الخطية الكسرية، y, x وكذلك $x^2=y^3=1$ تمثل علاقات على الزمرة PSL(2,Z).