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Toroidal Tokamak Equilibrium

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ABSTRACT. The inner and outer Magnetohydrodynamic equilibrium equations of the circular toroidal plasma Tokamak cross section solved to the first order of the inverse aspect ratio. The toroidal current density is of arbitrary profile. The shape of the conducting shell surrounding the plasma is determined.

In the traditional design approach, the confinement of the plasma is attempted by interposing magnetic fields between the plasma and the walls of the reaction chamber. This design attempts to improve plasma confinement by the closed toroidal Tokamak system.

Several authors (Lavel *et al.* 1971, Solovev and Shafranov 1970, Coppi *et al.* 1972) solved analytically the ideal MHD equilibrium of the circular and non-circular plasma cross section of the toroidal Tokamak with uniform profile of the toroidal current density.

In the present work, we solve analytically to the first order of inverse aspect ratio $\varepsilon = \frac{a}{R}$ the inner and outer MHD equilibrium equations of the toroidal Tokamak with finite beta (: the ratio kinetic and magnetic pressures). Here, a and R are the minor and major radii of the plasma, respectively. The plasma cross section is taken to be circular. Here, the toroidal current density profile is of arbitrary distribution. This investigation is essentially in the design studies of the Tokamak fusion reactor.

The equations of interest here are the ideal MHD equations:

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \vec{j}, \quad \text{and} \quad \vec{j} \times \vec{B} = \vec{\nabla} P \tag{1}$$

where $\vec{B}$, $\vec{J}$, and P are respectively the magnetic field, current density and scalar pressure.

By introducing the coordinate system (ϱ, θ, s) – (see Fig. 1), the expressions of the magnetic field, current density and MHD equilibrium equation have the form:

$$\vec{B} = f\vec{u} + \vec{u} \times \vec{\nabla} F \tag{2}$$

$$\vec{j} = (L_{op}F)\vec{u} + \vec{u} \times \vec{\nabla}(-f) \tag{3}$$

, and

$$L_{op}F + \frac{1}{2}\frac{df^2}{dF} + \left(1 - \frac{\varrho\cos\theta}{R}\right)^2 \frac{dP}{dF} = 0 \tag{4}$$

where

$$\vec{u} = \frac{\vec{e}_s}{\left(1 - \frac{\varrho\cos\theta}{R}\right)}, \quad \vec{\nabla} = \frac{\partial}{\partial\varrho}\vec{e}_\varrho + \frac{1}{\varrho}\frac{\partial}{\partial\theta}\vec{e}_\theta + \frac{1}{\left(1 - \frac{\varrho\cos\theta}{R}\right)}\frac{\partial}{\partial s}\vec{e}_s$$

$$L_{op} = \frac{1}{\varrho}\left(1 - \frac{\varrho\cos\theta}{R}\right)\left[\frac{\partial}{\partial\varrho}\frac{\varrho}{\left(1 - \frac{\varrho\cos\theta}{R}\right)}\frac{\partial}{\partial\varrho} + \frac{\partial}{\partial\theta}\frac{1}{\varrho\left(1 - \frac{\varrho\cos\theta}{R}\right)}\frac{\partial}{\partial\theta}\right]$$

$\vec{e}_s$ is a unit vector along the toroidal coordinates. The surface functions P and f are an arbitrary function of the equilibrium solution F. At this point, we would like to mention that the poloidal flux function ψ is related to the equilibrium solution F by the relation: $F = \dfrac{\psi}{2\pi R}$.

Here, we will assume that the functions $f_{(F)}$ and $P_{(F)}$ have the following forms:

$$P = P_0 + P_1 F \quad \text{and} \quad f^2 = f_0^2 - \frac{\mu^2}{a^2} F^2 \tag{5}$$

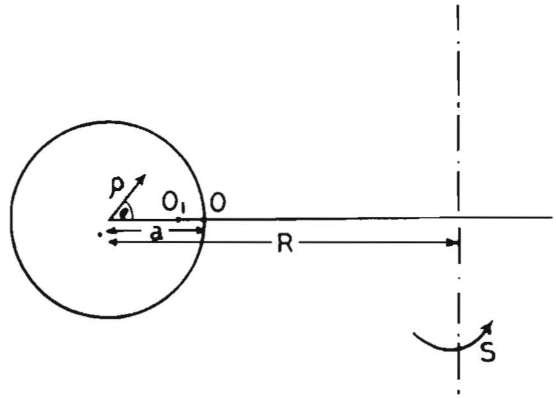

Fig. 1. Coordinate system.

where P_0, P_1, μ and f_0 are constants to be determined from the equilibrium solution. All these constants are characteristics of the plasma parameters.

The expression for the toroidal current is given by:

$$j_s(x,\theta) = \frac{L_{op}F}{\left(1 - \frac{a}{R}\,x\cos\theta\right)} = \left[\frac{\mu^2}{a^2}F - \left(1 - \frac{a}{R}\,x\cos\theta\right)^2 P_1\right]\left(1 - \frac{a}{R}\,x\cos\theta\right)^{-1} \tag{6}$$

where $x = \frac{\varrho}{a}$.

The equilibrium solution $F(x,\theta)$ of the equation (4) to the first order of inverse aspect ratio, under the assumption that $P(\varrho = a) = 0$, can be written in the form:

$$F(x,\theta) = F_0(x,\theta) + \varepsilon F_1(x,\theta) \tag{7}$$

where $F_0(x,\theta)$ is the solution to the zero order equilibrium equation:

$$\frac{\partial^2 F_0}{\partial x^2} + \frac{1}{x}\frac{\partial F_0}{\partial x} = \mu^2 F_0 - P_1 a^2 \qquad 0 \leqslant x \leqslant 1 \tag{8}$$

$$= 0 \qquad \text{for } x \geqslant 1 \tag{9}$$

This solution could be written in the form:

$$F_0^i(x,\theta) = -\frac{P_0}{P_1} + \widetilde{F}_0(E_\mu + E_0(x)) \qquad 0 \leqslant x \leqslant 1 \tag{10}$$

$$= F^{o}_{0e} + \alpha_0 \ln x \qquad \text{for } x \geqslant 1 \tag{11}$$

With

$$E_\mu = \left(\frac{1 - I_0}{2I_2}\right) \tag{12}$$

$$E_0(x) = \frac{I_0(\mu x) - 1}{2I_2} \tag{13}$$

where $\widetilde{F}_0$ ($= 2b_0I_2$), b_0, F^{o}_{0e} and α_0 are arbitrary constants. $I_i(\mu x)$ is the modified Bessel's function of the first kind and $I_i = I_i(\mu)$. The notations i and o are referred respectively to the values of the quantities in the inner (plasma) and outer (vaccum) regions. The function $F_1(x,\theta)$ is the solution of the first order equilibrium equation, namely:

$$\frac{\partial^2 F_1}{\partial x^2} + \frac{1}{x}\frac{\partial F_1}{\partial x} + \frac{1}{x^2}\frac{\partial^2 F_1}{\partial \theta^2} = \mu^2 F_1 + (2a^2P_1x - b_0\mu I_1(\mu x))\cos\theta$$

$$\text{for } 0 \leqslant x \leqslant 1 \tag{14}$$

$$= -\left(\frac{\alpha_0 \cos\theta}{x}\right) \quad \text{for } x \geqslant 1 \tag{15}$$

The solution of equations (14) and (15) can be respectively written as:

$$F^{i}_1(x,\theta) = \widetilde{F}_0 E_1(x) \cos\theta \qquad \text{for } 0 \leqslant x \leqslant 1 \tag{16}$$

$$F^{o}_1(x,\theta) = \left[-\frac{\alpha_0}{2} x \ln x + \alpha_1\left(x - \frac{1}{x}\right)\right]\cos\theta \text{ for } x \geqslant 1 \tag{17}$$

where

$$E_1(x) = \left[\left(\frac{2P_1a^2}{\mu^2 b_0}\right)\left(\frac{1}{2I_2}\right)\left(\frac{I_1(\mu x)}{I_1} - x\right) + \frac{I_0}{4I_2}\left(\frac{I_1(\mu x)}{I_1} - \frac{xI_0(\mu x)}{I_0}\right)\right] \tag{18}$$

where α_1 is an arbitrary constant.

The average pressure $\langle P \rangle \left(= \frac{1}{V}\int P dV\right)$ is given by: $\langle P \rangle = -P_1 b_0 I_2(\mu)$ where V is the volume bounded by F = constant. The parameters: $\beta = \left(\frac{2\langle P \rangle}{B^2_s\,(x = 1)}\right)$, poloidal beta $\beta_P\left(=\dfrac{2\langle P\rangle}{B_\theta^2\,(x=1)}\right)$, the total toroidal current $I_s\left(=a^2\int_0^1 x dx\int_0^{2\pi} d\theta j_s(x,\theta)\right)$, the inverse of the safety factor $\dfrac{1}{\hat{q}(a)}\left(=\dfrac{RB_\theta\,(x=1)}{aB_s\,(x=1)}\right)$ for an equivalent cylinder of radius a and the internal inductance $l_i\left(=\dfrac{1}{\pi B_\theta^2\,(x=1)}\int_0^1 dx\int_0^{2\pi} d\theta(B_\theta^2 x)\right)$ per unit length of a plasma ring defined by Solovev and Shafranov (1970), are respectively given by:

$$\left.\begin{aligned} &\beta=-\frac{2P_1 b_0 I_2}{f^2\,(x=1)} \quad , \quad \beta_P=\frac{\beta}{\varepsilon^2\left(\dfrac{1}{\hat{q}(a)}\right)^2} \\ &I_s=2\pi b_0\mu I_1 \quad , \quad \frac{1}{\hat{q}(a)}=\frac{1}{\varepsilon}\frac{1}{f\,(x=1)}\left(\frac{b_0\mu}{a}\right) I_1 \\ &l_i = 1-\left(\frac{I_0 I_2}{I_1^2}\right) \end{aligned}\right\} \tag{19}$$

where B_θ and B_s are respectively the poloidal and toroidal components of the magnetic field $\vec{B}$.

By using the above parameters, we could write respectively the functions $E_1(x)$ and $E(x,\theta)$, the pressure $P(x,\theta)$ and the inner equilibrium solution F^i in the following forms:

$$E_1(x)=\left(\frac{I_0}{4I_2}\right)\left(\frac{I_1(\mu x)}{I_1}-\frac{xI_0(\mu x)}{I_0}\right)+\beta_P\left(\frac{I_1^2}{2I_2^2}\right)\left(x-\frac{I_1(\mu x)}{I_1}\right) \tag{20}$$

$$E(x,\theta) = E_\mu + E_0(x) + \varepsilon E_1(x)\cos\theta \tag{21}$$

$$F^i(x,\theta) = -\frac{P_0}{P_1} + \tilde{F}_0 E(x,\theta) \tag{22}$$

$$P(x,\theta) = \tilde{F}_0 P_1 E(x,\theta) \tag{23}$$

with

$$\tilde{F}_0 = (2b_0 I_2) = \left(\frac{a\varepsilon f\,(x=1)}{2}\right)\left(\frac{1}{\hat{q}(a)}\ \frac{4I_2}{\mu I_1}\right) \tag{24}$$

It is clear that the function $E(x,\theta)$ and the pressure $P(x,\theta)$ vanish on the plasma boundary $x = 1$. Also, we have

$$F'(x = 1, \theta) = -\frac{P_0}{P_1}$$

Here, we will introduce the parameter ν defined as:

$$\frac{l_i}{2} + \beta_P = \left(\frac{\nu}{\alpha/R}\right) \tag{25}$$

The position of the magnetic axis whose coordinates are assumed to be $= \pi$ and $x = x_m$ is determined from the condition $\frac{\partial E\,(x = x_m, t = \pi)}{\partial x} = 0$. For the case of slightly non-uniform toroidal current density profile, the expression for x_m is given by:

$$x_m = \left(\frac{a}{R}\right)\left(\frac{I_1}{\mu I_2}\right)\left[\frac{I_2^2}{2I_1^2} + \left(\frac{2I_1}{\mu} - 1\right)\beta_P\right] \tag{26}$$

The limiting values of beta β_{Pmax} for equilibrium is defined as the value of beta at which a second magnetic axis on the plasma to boundary will appear. This value is given by:

$$\beta_{Pmax} = \left(\frac{a}{R}\right)^2 \frac{1}{(\hat{q}(a))^2}\left[\frac{R}{a} - \frac{1}{2}\left(1 - \frac{I_0 I_2}{I_1^2}\right)\right] \approx \left(\frac{a}{R}\right)\frac{1}{(\hat{q}(a))^2} \tag{27}$$

In other words,the condition for the appearance of a second magnetic axis is given by $\nu < 1$ or $\beta < \beta_{Pmax}$.

The toroidal current density $j_s(x,\theta)$ for the class of equilibrium under consideration is given from (6) and (20-22) as:

$$j_s(x,\theta) = \left\{\left(j_0 + \frac{\mu^2}{a^2}\tilde{F}_0 E_0(x)\right) + \varepsilon\left(\frac{\mu^2}{a^2}\tilde{F}_0 E_1(x) + 2xP_1\right)\cos\theta\right\}\left(1 - \varepsilon x \cos\theta\right)^{-1} \tag{28}$$

where j_0 is the central toroidal current density, which is given by:

$$j_0 = \left[-\left(\frac{P_0}{P_1}\frac{\mu^2}{a^2} + P_1\right) + \frac{\mu^2}{a^2}\tilde{F}_0\left(\frac{1 - I_0}{2I_2}\right)\right] \tag{29}$$

In this case, the parameter $\frac{1}{\hat{q}(a)}$ could be expressed as

$$\frac{1}{\hat{q}(a)} = \frac{1}{\bar{q}(a)}\left(\frac{2I_1}{\mu}\right)\left(1 + O(\varepsilon^2)\right) \tag{30}$$

where

$$\frac{1}{\bar{q}(a)} = \left(\frac{Rj_0}{2f\,(x = 1)}\right)\left(1 + O(\varepsilon^2)\right) \tag{31}$$

By using (24), (3) and (31) into (28), we can rewrite expression for $j_s(x,\theta)$ as follows:

$$j_s(x,\theta) = j_0\left\{I_0(\mu x) + \varepsilon\left[2I_2E_1(x) - \left(\frac{I_1}{I_2}\right)^2\beta_P\right]\cos\theta\right\}\left(1 - \varepsilon x\cos\theta\right)^{-1} \tag{32}$$

Figure 2 shows to the zero order of approximation the toroidal current distribution over the plasma cross section of the Tokamak. This distribution depends on the value of μ.

At a point (defined by $x = 1$ and $\theta = 0$) on the plasma boundary, the longitudinal current density vanishes when $\beta_P = \beta_{Pc}$, where

$$\beta_{Pc} = \left(\frac{R}{a}\,\frac{I_0I_2}{I_1^2}\right) \tag{33}$$

At this point, the Tokamak equilibrium can not be maintained. The current density will reverse at a point (defined by: $x = x_0$ and $\theta = 0$) if $\beta_P > \beta_{Pc}$. In other words, the consideration of the first order of approximation leads to the inverse of the toroidal current density with all profiles inside the plasma.

The inner equilibrium function $E(x,\theta)$ for the case of slightly non-uniform toroidal current has the form:

$$E(x,\theta) = \frac{\mu^2}{8I_2}\,(1 - x^2)(-1 + \nu_m x\cos\theta) \tag{34}$$

where

$$\nu_m = \left(\nu - \frac{a}{R}\,\delta_\mu\right)\left(\frac{\mu I_1}{4I_2}\right) \quad \text{and} \quad \delta_\mu = \frac{1}{2}\left(1 - \frac{I_0I_2}{I_1^2}\right) - \frac{1}{4}\left(\frac{4I_2}{\mu I_1}\right)^2$$

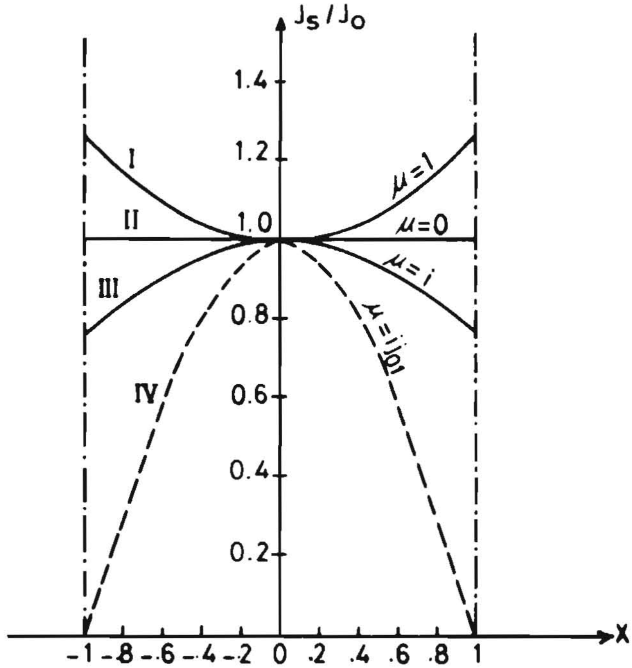

Fig. 2. Various toroidal current density distribution (j_s/j_0) profiles over the whole plasma cross section of the toroidal Tokamak configuration with ε = 0.2.

The hyperbolic point of the separatrix ($\nu > 1$) is $x_s = 1$ and $\theta = \cos^{-1}\frac{1}{\nu}$, at which

$E = E_s = 0, \quad P = P_s = 0 \quad \text{and} \quad F = F_s = -\frac{P_0}{P_1}$.

While for the case of uniform distribution profile this function becomes:

$$E(x,\theta) = (1 - x^2)(-1 + \nu x \cos\theta) \tag{35}$$

The form of the solution $F^i(x,\theta)$ near the magnetic axis (x_m,π) could be written in the form:

$$F(x,\theta) = F_m + (A + B\cos 2\theta)\varrho^2 + (C\cos\theta + D\cos 3\theta)\varrho^3 \tag{36}$$

where

$$F_m = -\frac{P_0}{P_1} + \widetilde{F}_0\left(\frac{\mu^2}{8I_2}\right)E_{m0},\ E_{m0} = 1 + x_m^2 + \nu x_m - \nu x_m^3 \tag{37}$$

$$A = -C_0(1 + 2\nu_m' x_m),\ B = -\left(\frac{C_0}{a}\right)\nu_m,\ C = -\left(\frac{C_0}{a}\right)\nu_m,\ D = 0$$

and

$$C_0 = \left(\frac{\widetilde{F}_0}{a^2}\right)\frac{\mu^2}{8I_2}$$

The constants F_{0e}^o, α_0 and α_1 are determined from the following pressure balance conditions on the plasma boundary: $(B_\theta^i)^2 + (B_s^i)^2 = (B_\theta^o)^2 + (B_s^o)^2$, assuming that the pressure vanishes at x = 1. The result of calculation is:

$$F_{0e}^o = F_s = F_0^i\left(x = 1, \theta = \cos^{-1}\frac{1}{\nu}\right) = -\frac{P_0}{P_1},\ \left(\frac{\alpha_0}{a}\right) = \varepsilon_0 f_0 \frac{1}{\hat{q}(a)},$$

and $\left(\frac{2\alpha_1}{a}\right) = \left(\frac{\alpha_0}{a}\right)\left(\frac{1}{2} - \frac{R\nu}{a}\right)$

In this case, the outer equilibrium takes the form:

$$F^o(x,\theta) = F_{0e}^o + \left(a\varepsilon_0 f_0 \frac{1}{\hat{q}}\right)\left\{\ln x + \varepsilon\left[-\frac{x}{2}\ln x + \frac{1}{2}\left(\frac{1}{2} - \frac{R\nu}{a}\right)\left(x - \frac{1}{x}\right)\right]\cos\theta\right\} \tag{38}$$

The deformation of the magnetic surface near $\varrho = b$ can then be computed by substituting $\varrho = b + \Delta \cos\theta$ in the solution (37), where b is the radius of the conducting shell – (see Fig. 3), and Δ is the plasma displacement.

Finally, the MHD equilibrium problem of a toroidal Tokamak configuration with a circular plasma cross section and a toroidal current density of arbitrary distribution solved analytically. We found that the limiting value of Beta for the reverse of the toroidal current inside the plasma is given by $\beta_P < \beta_{P_c}$, while that for the appearance of both of a second magnetic axis on the plasma boundary and a separatrix inside the plasma are given by $\beta < \beta_{P_{max}}$. MHD equilibrium can be achieved by shaping the cross section of a conducting shell so as to make it can coincide with one of the magnetic surfaces in vacuum. The advantage of this study is to determine the optimum toroidal current profile for equilibrium, which should be considered when the fusion reactor will be designed.

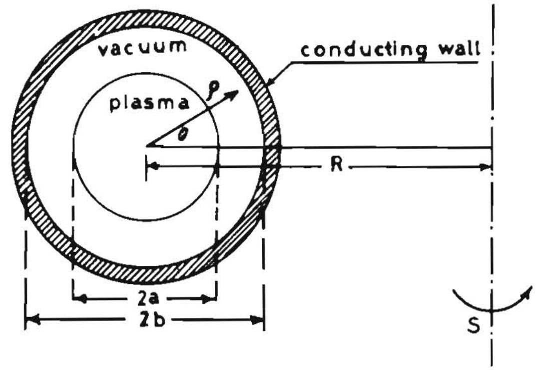

Fig. 3. A circular plasma of radius a is confined by an outer perfectly conducting shell of radius b.

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