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Circular Braids

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ABSTRACT. Coxeter-type generalisations of the braid groups are introduced, and their structure is elucidated in the case when the associated graph is a circle.

1. Consider presentations of the form

$$G = \langle\, x_i,\ 1 \leqq i \leqq n \,|\, r_{ij},\ 1 \leqq i \leqq j \leqq n \rangle, \tag{1}$$

where each r_{ij} is either

$$[x_i,x_j] := \bar{x}_i\bar{x}_jx_ix_j \text{ or } (x_i,x_j) := \bar{x}_i\bar{x}_j\bar{x}_ix_jx_ix_j,$$

a commutator or a braid relation, respectievely, (where $\bar{x} := x^{-1}$, for $x \varepsilon G$). Such a group is clearly described by the isomorphism-type of its *graph* Γ, which has n vertices numbered from 1 to n, with an edge between i and j if and only if $r_{ij} = (x_i, x_j)$. The following points are worthy of note.

(i) The adjunction of relators $\{x_i^2 | 1 \leqq i \leqq n\}$ yields a class of Coxeter groups.

(ii) Since G is the direct product of the groups corresponding to the components of Γ, it can safely be assumed that Γ is connected.

(iii) The case when Γ is an interval yields the familiar braid group B_{n+1} of Artin (1925) (for a more recent survey, see Magnus 1974).

As a first step in the study of this class of groups, the aim of this article is to obtain an algebraic description of the group $\hat{B}_n$ of *circular braids*, defined by (1) when Γ is a circle. The method is purely algebraic, though it depends for its inspiration on the elementary combinatorial ideas outlined in the next section.

2. In terms of the familiar picture, $\hat{B}_n$ may be derived from B_n by declaring that the crossing of the 1 and n strings be independent of the others. This yields in a natural way, a split epimorphism $\theta : \hat{B}_n \rightarrow B_n$, and so the problem reduces to describing Ker θ as a B_n-group. Now the independence of this crossing is guaranteed by the insertion of an extra string O between it and the crossings of adjacent strings. In algebraic terms, this yields a homomorphism from $\hat{B}_n$ to the group P_{n+1} of O-pure (n + 1)-braids. Now it is well known (Chow 1948) that P_{n+1} is a split extension of B_n by a free group $F_n = \langle a_0, \ldots, a_{n-1} | \rangle$, with action given by the usual embedding of B_n in Aut F_n (see (**3**) below).

This is summarised in the commutative diagram

$$\begin{tikzcd}
 & & \hat{B}_n \arrow[d, "\phi"'] \arrow[dr, "\theta"] & & \\
1 \arrow[r] & F_n \arrow[r, "\mathrm{inc}"] & P_{n+1} \arrow[r] & B_n \arrow[r] & 1,
\end{tikzcd}$$

where the row is exact. Under the assumption that $\mathrm{Ker}\phi_n\,\mathrm{Ker}\theta = 1$, the problem reduces to a description of $(\mathrm{Ker}\theta)\phi = F_n \cap \mathrm{Im}\phi$, and this turns out to be the B_n-normal closure F of the single element $a_0^{-1}a_{n-1} \varepsilon F_n$. An easy calculation shows that F is just the subgroup of F_n consisting of all words of exponent-sum 0, and so the programme is now as follows:

(i) find Schreier generators for F,
(ii) compute the action of B_n on these,
(iii) form the split extension of B_n by F, and
use Tietze transformations to reduce the resulting presentation to that of B_n, which culminates in the following result.

Theorem. *The group* $\hat{B}_n$ *is a split extension of* B_n *by a free group* F *of countably infinite rank, where the action is given by regarding* F *as the subgroup of* F_n *consisting of all words of total exponent-sum zero.*

In the proof, which occupies the remainder of the article, all the steps but one are straightforward, and this is concentrated into a lemma (§ **6**), which may be of independent interest.

3. The Schreier transversal $\{a_0^k | k \varepsilon \mathbf{Z}\}$ for F in F_n yields in the following set of free generators for F:

$$\{c_{k,i} := a_0^k a_i \bar{a}_0^{k+1} | k \varepsilon \mathbf{Z},\ 1 \leqq i \leqq n-1\}.$$

Using a succession of Nielsen transformations, this can be replaced by a set $\{d_{k,i} | k \in \mathbf{Z}, 1 \leqq i \leqq n-1\}$, there

$$d_{k,i} = \begin{cases} c_{0,i} \ldots c_{k,i} = a_i^{k+1} \bar{a}_0^{k+1}, & k \geqq 0, \\ c_{k,i} \ldots c_{-1,i} = a_0^k \bar{a}_i^k, & k < 0. \end{cases}$$

Finally, let

$$(k,i) = \left\{ \begin{matrix} d_{k-1,i}, & k > 0 \\ \bar{d}_{k,i}, & k < 0 \end{matrix} \right\} = a_i^k \bar{a}_0^k \tag{2}$$

for all non-zero values of k, and interpret $(o, i) = e$ for convenience $(1 \leqq i \leqq n-1)$.

The action on B_n of F_n is given by $(o \leqq i \leqq n-1, 1 \leqq j \leqq n-1)$

$$a_i \cdot x_j = \begin{cases} a_{i+1} & , \text{if } j = i+1, \\ \bar{a}_i a_{i-1} a_i, & \text{if } j = i \\ a_i & , \text{otherwise}, \end{cases} \tag{3}$$

and this yields the following action of B_n on the generators (2) of F:

$$(k,i)\sigma_j = (k,i), \; j \neq 1, i, i+1, \tag{4}$$

$$(k,i)\sigma_{i+1} = (k,i+1), \tag{5}$$

$$(k,i)\sigma_i = (-1,i)\overline{(-1,i-1)}(k-1,i-1)\overline{(k-1,i)}(k,i), \; i \neq 1, \tag{6}$$

$$(k,i)\sigma_1 = (k,i)\overline{(k,1)}, \; i \neq 1, \tag{7}$$

$$(k,1)\sigma_1 = (-1,1)\overline{(k-1,1)}. \tag{8}$$

Thus, the desired split extension is obtained by adjoining the generators (2) and relations (4)–(8) to the usual presentation of B_n.

4. The first step is to reduce the relations (4)–(8) to a more manageable form ((4″), (8″), (7″) below), and this is done in four stages.

(i) The relations (6) are superfluous: induct on $i \geqq 2$. First of all,

$$(k,2)x_2 = [(k,2)x_1 \cdot (k,1)]x_2, \text{ by } (7),$$

$$= (k,1)x_2x_1x_2 \cdot (k,1)x_2, \text{ by } (5),$$

$$= (k,1)x_1x_2x_1 \cdot (k,1)x_2, \text{ by a braid relation,}$$

$$= [(-1,1) \cdot \overline{(k-1,1)}]x_2x_1 \cdot (k,1)x_2, \text{ by (8),}$$

$$= (-1,2)\sigma_1 \cdot \overline{(k-1,2)}\sigma_1 \cdot (k,2) \text{ by (5),}$$

$$= (-1,2) \cdot \overline{(-1,1)} \cdot (k-1,1) \cdot \overline{(k-1,2)} \cdot (k,2), \text{ by (7),}$$

as required. For $i > 2$,

$$\begin{aligned}(k,i)\sigma_i &= (k,i)\sigma_{i-1}\sigma_i, \text{ by (4)}\\ &= (k,i-1)\sigma_i\sigma_{i-1}\sigma_i, \text{ by (5)}\\ &= [(-1,i-1) \cdot \overline{(-1,i-2)} \cdot (k-1,i-2) \cdot \overline{(k-1,i-1)} \cdot (k,i-1)]\sigma_i\sigma_{i-1},\\ &\quad \text{by induction,}\\ &= [(-1,i) \cdot \overline{(-1,i-2)} \cdot (k-1,i-2) \cdot \overline{(k-1,i)} \cdot (k,i)]\sigma_{i-1}, \text{ by (4) and (5),}\\ &= (-1,i) \cdot \overline{(-1,i-1)} \cdot (k-1,i-1) \cdot \overline{(k-1,i)} \cdot (k,i), \text{ by (4) and (5),}\end{aligned}$$

as required.

(ii) The relations (4) are all consequences of those with $i = 1$, namely

$$(k,1)x_j = (k,1), \ j \geqq 3; \tag{4'}$$

for, by (5), $(k, i)x_j = (k, 1)x_2 \ldots x_ix_j$, and this is (k, i) when $j > i + 1$ (by the relations of B_n), while for $2 \leqq j \leqq i - 1$, the right-hand side is $(k, 1)x_{j+1}\, x_2 \ldots x_i = (k, i)$ by (4′).

(iii) The relations (7) are all consequences of those with $i = 2$, namely (using (5))

$$(k,1)x_2x_1 = (k,1)x_2 \cdot \overline{(k,1)}. \tag{7'}$$

$$\begin{aligned}\text{For } i \geqq 2,\ (k,i)x_1 &= (k,1)x_2 \ldots x_ix_1, \text{ by (5),}\\ &= (k,1)x_2x_1x_3 \ldots x_i, \text{ by the relations of } B_n,\\ &= [(k,1)x_2 \cdot \overline{(k,1)}]x_3 \ldots x_i, \text{ by (7′),}\\ &= (k,i)\overline{(k,1)}, \text{ by (5) and (4′).}\end{aligned}$$

(iv) Using (5), the generators (k, i) for $i \geqq 2$ can now be eliminated by a Tietze transformation. Writing $(k, 1) = k$: $k \varepsilon \mathbf{Z}$, we have reduced the presentation to that of B_n, augmented by generators $\mathbf{Z}$ and relators

$$k^{x_1} = -1) \cdot \overline{k-1}, \tag{8''}$$

$$k^{x_j} = k, j \geqq 3, \tag{4''}$$

$$k^{x_2x_1} = k^{x_2} \cdot \bar{k}. \tag{7''}$$

Now (8″) simply asserts that the generators $k \neq 1$ are superfluous (note that $O = e$ and $1^{x_1} = -1$, in particular). Next, the relations (4″) are all consequences of those with $k = 1$, namely

$$1^{x_j} = 1, 3 \leqq j \leqq n - 1, \tag{4'''}$$

because of (8″) and the relations of B_n. Finally, all the relations (7″) are consequences of those with $k = \pm 1$ (together with (8″) and (4″)), namely

$$1^{x_2x_1} = 1^{x_2} \cdot \bar{1}, \tag{7'' + 1}$$

$$1^{x_1x_2x_1} = 1^{x_1x_2}\, \bar{1}^{x_1}. \tag{7'' - 1}$$

The proof of this is deferred to § 6, and we show first that the resulting presentation yields $\hat{B}_n$.

5. We first examine the effect of the following change of generators:

$$\left.\begin{aligned} a &= x_2 \\ b &= x_1 \cdot 1 \\ c &= \bar{x}_2 x_1 x_2 \end{aligned}\right\} \quad \left\{\begin{aligned} \sigma_2 &= a, \\ \sigma_1 &= ac\bar{a}, \\ 1 &= a\overline{ca}b, \end{aligned}\right.$$

adopting for convenience the notation $x \sim y$ means $[x, y] = e$ and $x \approx y$ means $(x, y) = e$. Now the relations of B_n not involving x_1 or x_2 remain unaltered, and it is easy to check that

$$x_1 \approx x_2, (7'' + 1), (7'' - 1)$$

pass to

$$c \approx a, a \approx b, b \approx c \tag{9}$$

respectively. Also, the relations r_{1j} and r_{2j} ($j \geqq 3$) of B_n, (4″′) pass to

$$a c \bar{a} \sim x_3, \ldots, x_{n-1}, \tag{10}$$

$$a \approx x_3,\ a \sim x_4, \ldots, x_{n-1}, \tag{11}$$

$$a \overline{ca} b \sim x_3, \ldots, x_{n-1}, \tag{12}$$

respectively.

The presentation now reads

$$\langle a,b,c,x_3,\ldots,x_{n-1} \mid (9)\text{-}(12),\ r_{ij} = e,\ 3 \leqq i < j \leqq n-1 \rangle,$$

clearly equal to $\hat{B}_3$ when $n = 3$. For general n, one further change of generator is required, namely,

$$\begin{cases} a' = a^{x_3 \ldots x_{n-1}} = x_{n-1}{}^{x_{n-2} \ldots x_3 a}, \\ a = a'^{x_{n-1} \ldots x_3}, \end{cases}$$

where the second equation follows from $r_{ij} = e$ and $a \approx x_3$.

It remains to show that these relations pass to those of $\hat{B}_n$ on a', b, c, x_3, ..., x_{n-1}, which we call s_{ij}, $0 \leqq i \leqq n-1$. In view of (10), we have

$$(12) \rightarrow s_{1j},\ 3 \leqq j \leqq n-1$$

Because of (11),

$$(10) \rightarrow s_{2j},\ 4 \leqq j \leqq n-1,\ s_{02}.$$

A simple calculation shows that

$$(11) \rightarrow s_{0j},\ 3 \leqq j \leqq n-2,\ s_{0.n-1}.$$

Finally,

$$(9) \rightarrow s_{23},\ s_{01},\ s_{12},$$

and we have arrived at $\hat{B}_n$.

6. Returning to the deferred step of the proof, the crucial observation is that (7″) is a braid relation; in fact, writing $x = x_1$, $y = x_2$ and using $x \approx y$, it becomes

$$r_k : y \approx kx,\ k \,\varepsilon\, \mathbf{Z}.$$

Putting $d = -1$, this asserts that $y \approx dx, x, xd$ (taking $k = -1, 0, 1$). Furthermore, (8″) can be written in either of the forms

$$(k - 1)x = \overline{kx} \cdot xdx, \text{ or}$$

$$(k + 1)x = xdx \cdot \overline{kx}.$$

Taking $t = kx$, the following lemma thus asserts that

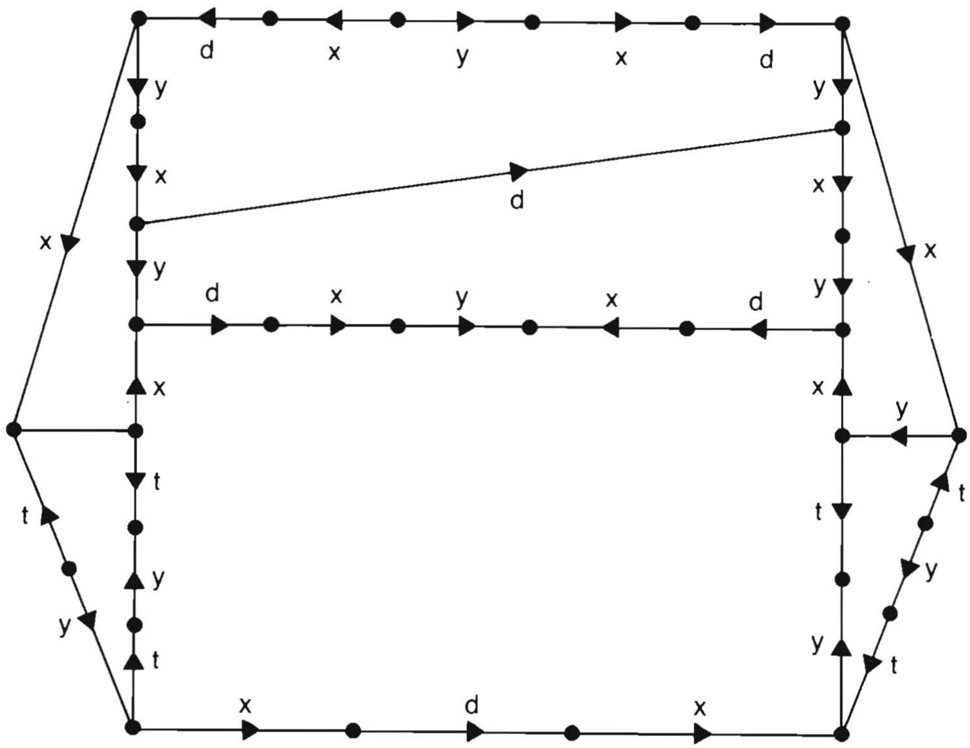
$$r_{-1}, r_0, r_1, r_k \Rightarrow (r_{k-1} \Leftrightarrow r_{k+1}),$$

which affords an inductive proof that all the r_k are consequences of $r_{\pm 1}$ and $y \approx x$.

Lemma. If y, d, x, t are elements of any group, such that $y \approx dx, x, xd$ and t, then

$$y \approx \bar{t}xdx \Leftrightarrow y \approx xdx\bar{t}.$$

Proof.



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