Arab Gulf J. scient. Res. **2** (2), pp. 561-582 (1984)

Deflection Surface of a Symmetrically Loaded Thin Circular Annulus on Multipoint Supports

W.A. Bassali

Faculty of Science, University of Kuwait, Kuwait

ABSTRACT. An infinite series solution is obtained for the small deflection of a thin circular annular plate subject to symmetrical loading distributed over the entire plate and having several point supports which are situated at equal distances apart on a single concentric circle. Limiting forms of the derived solution are investigated.

1. Introduction

Thin circular plates with or without central holes are structures which have considerable applications in engineering design and problems dealing with their transverse flexure have been studied both theoretically and experimentally by many investigators. Bassali (1957) applied complex variable methods to establish exact solutions in closed forms for the deflections of a thin circular plate supported at interior or boundary points and normally loaded over an eccentric circle, the load being symmetric with respect to the centre of the circle. Symmetrical bending of a circular plate supported at points regularly distributed along the circumference of a concentric circle is discussed as a special case in the same paper. In two papers, Kirstein and Wooley (1966, 1968) obtained the deflections of thin circular plates under symmetrically distributed loading over a concentric circle, the plate being supported on equally spaced point supports on another concentric circle lying either within or outside the loaded area. Real variable methods were used by Williams and Brinson (1974) to study the same problems. Their method of solution is that of superposition of relevant known results and several examples are worked out in detail. Comparison of theory with experimental data leads to favorable conclusions. In a recent paper (Chantaramungkorn *et al.* 1973), a general solution is presented for the lateral deflection of a circular plate supported at its periphery by an arbitrary number of equally spaced columns and acted upon by an eccentric concentrated load having various locations. The results of Kirstein and Wooley (1966, 1968), Williams and Brinson (1974) and Chantaramungkorn *et al.* (1973) are essentially particular cases of the general problem treated by Bassali (1957). In two papers (Bassali and Gorgui, 1960a and b), explicit expressions were found for the deflection at any point of a circular annular plate under various combinations of edge conditions when the plate is normally loaded by (i) general line loading along the circumference of a concentric circle (ii) a concentrated load (iii) a concentrated couple (iv) general lateral loadings distributed over the entire circular ring plate. Bassali (1960) also derived infinite series solutions for the deflections of a circular annular plate under any normal system of concentrated forces or concentrated couples in equilibrium, the edges of the annulus being free. Axisymmetrical bending of circular ring plates was also studied by Heap (1968) who gave formulae and curves for the stresses and deflections with a variety of constraint conditions at the inner and outer edges, the plate being under the action of a line load on a concentric circle, a problem which was extensively discussed by Bassali and Gorgui (1960a). Other investigations connected with normally loaded circular ring plates can be found in references (Bassali and Gorgui 1960a and b, and Bassali 1960).

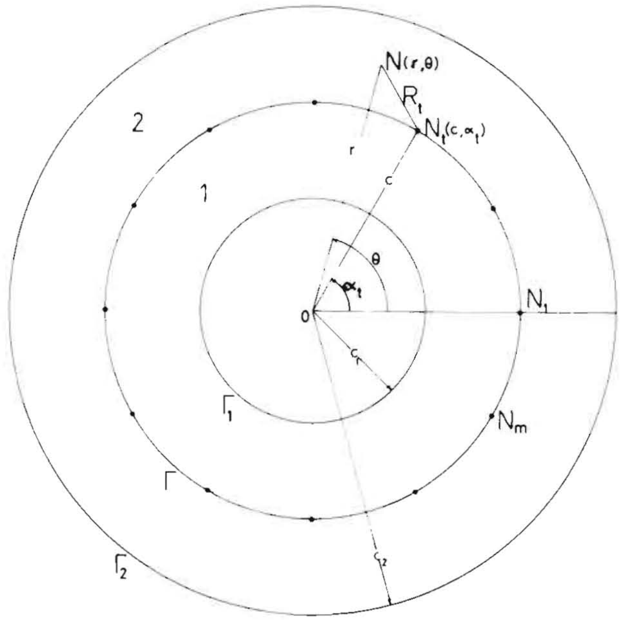
This paper is concerned with the derivation of formulae for the deflection surface of a symmetrically loaded thin circular annular plate supported at equally spaced points on a concentric circle. The case in which the radius of the hole tends to zero is studied and for uniform loading the obtained expression for the deflection agrees with that of Williams and Brinson (1974, p. 438). Limiting cases in which the diameter of the support circle becomes as large as the diameter of the plate or as small as the diameter of the hole are also discussed. The solutions obtained are based on the classical Poisson-Kirchhoff theory of small deflections of thin isotropic plates and are exact within the assumptions underlying this theory.

2. Mathematical Formulation of the Problem

The problem to be solved consists of determining the deflection surface of a thin constant thickness circular ring plate bounded by an inner circle Γ_1 of radius c_1 and an outer circle Γ_2 of radius c_2. Both edges of the plate are assumed to be free and the plate is subject to symmetrical normal pressure distributed over the entire plate and is supported by m concentrated forces located at equal distances apart on a concentric circle Γ of radius c. Since it is advantageous to work with dimensionless quantities we introduce the notations

$$\lambda_1 = c_1/c, \qquad \lambda_2 = c/c_2, \qquad \lambda = c_1/c_2 = \lambda_1\lambda_2. \tag{1}$$

The small transverse displacement w is measured from the plane of the supports.



Relative to this plane, downward deflections are positive and upward deflections are negative. If (r,θ) are the polar coordinates of any point N in the mid-plane , then the deflection $w(r,\theta)$ at this point satisfies the biharmonic equation

$$\nabla^4 w(r,\theta) = p(r,\theta)/D, \tag{2}$$

where $\nabla^2 = d^2 + r^{-1}d + r^{-2}d'^2, \quad d = \partial/\partial r, \quad d' = \partial/\partial\theta,$ (3)

D is flexural rigidity of the plate and $p(r,\theta)$ is the normal load intensity at the point N. We shall consider the case

$$p(r,\theta) = p_0 r^{s-2}, \tag{4}$$

where p_0 and s are constants. For $s = 2$ we have uniform loading over the entire plate.

In polar coordinates the formulae for the bending moments M_r, M_θ, twisting moment $M_{r\theta}$ and shearing forces Q_r, Q_θ per unit length of the middle surface of the plate are (Timoshenko and Woinowsky-Krieger 1959, p. 283):

$$M_r = -D(d^2 + \nu r^{-1}d + \nu r^{-2}d'^2)w, \tag{5a}$$

$$M_\theta = -D(\nu d^2 + r^{-1}d + r^{-2}d'^2)w, \tag{5b}$$

$$M_{r\theta} = (1 - \nu)Dr^{-1}(d - r^{-1})d'w, \tag{5c}$$

$$Q_r = -Dd(\nabla^2 w), \qquad Q_\theta = -Dr^{-1}d'(\nabla^2 w), \tag{5d}$$

where ν is Poisson's ratio for the material of the plate. The intensity V_r of the vertical reaction at either edge is furnished by

$$\begin{aligned} V_r &= Q_r - r^{-1}\partial M_{r\theta}/\partial\theta \\ &= [d^3 + r^{-1}d^2 - r^{-2}d + r^{-2}\{(2-\nu)d + (\nu - 3)r^{-1}\}d'^2]w, \end{aligned} \tag{6}$$

where $r = c_1$ or c_2 according as we are dealing with the inner edge Γ_1 or outer edge Γ_2, respectively. The conditions for the two edges to be free are

$$(M_r)_{r = c_j} = 0, \tag{7}$$

$$(V_r)_{r = c_j} = 0, \tag{8}$$

where $j = 1, 2$, M_r and V_r are given by (5a) and (6), respectively.

3. Method and Solution

Let $z_t = ce^{i\alpha_t}$ $(t = 1, 2, \ldots, m)$ be the m points of support N_t where

$$\alpha_t = (t-1)\alpha, \qquad \alpha = 2\pi/m, \qquad m \geqslant 2, \qquad t = 1, 2, \ldots, m. \tag{9}$$

If P_t is the concentrated reaction, measured positively downwards, at the point N_t then

$$P_t = -P_0/m \ (t = 1, 2, \ldots, m), \tag{10}$$

where P_0 is the total load on the plate which is given by

$$P_0 = 2\pi \int_{c_1}^{c_2} rp(r)dr. \tag{11}$$

Substitution from (4) in (11) yields

$$P_0 = 2\pi p_0(c_2^s - c_1^s)/s \quad \text{for } s \neq 0, \qquad P_0 = 2\pi p_0 \ln(c_2/c_1) \quad \text{for } s = 0. \tag{12}$$

It is known that the singular part of the deflection w at any point $N(r,\theta)$ near the point of application of the concentrated force P_t is

$$w^t_{sin} = \frac{P_t}{8\pi D} R_t^2 \ln R_t, \tag{13}$$

where R_t is the distance NN_t which is given by

$$R_t^2 = r^2 + c^2 - 2cr\cos\theta_t, \qquad \theta_t = \theta - \alpha_t. \tag{14}$$

Let 1 refer to the region $c \geqslant r \geqslant c_1$ between Γ_1 and Γ and 2 refer to the region $c_2 \geqslant r \geqslant c$ between Γ and Γ_2. See Fig. 1. Assuming that $\varrho = r/c$ we have (Bassali and Gorgui 1960a, p. 85)

$$R_t^2 \ln R_t = c^2\Big[\varrho^2 + (1+\varrho^2)\ln c - \varrho(1 + 2\ln c + 1/2\,\varrho^2)\cos\theta_t + \sum_2^\infty \frac{\varrho^n}{n}\Big(\frac{1}{n-1} - \frac{\varrho^2}{n+1}\Big)\cos n\theta_t \qquad (\varrho \leqslant 1), \tag{15a}$$

$$R_t^2 \ln R_t = c^2\Big[\;1 + (1+\varrho^2)\ln r - \varrho(1 + 2\ln r + 1/2\,\varrho^{-2})\cos\theta_t + \sum_2^\infty \frac{\varrho^{-n}}{n}\Big(\frac{\varrho^2}{n-1} - \frac{1}{n+1}\Big)\cos n\theta_t \qquad (\varrho \geqslant 1), \tag{15b}$$

It is easily proved that

$$\sum_{t=1}^{m} \cos n\theta_t \begin{cases} = m & \text{if } n = 0,\\ = m\cos n\theta & \text{if } n = m, 2m, 3m, \ldots\\ = 0 & \text{if } n \neq m, 2m, 3m, \ldots \end{cases} \tag{16}$$

From (10), (13), (15) and (16) we deduce that

$$\sum_{t=1}^{m} w^t_{sin} = -\frac{P_0}{8\pi m D}\sum_{t=1}^{m} R_t^2 \ln R_t$$

$$= -\frac{P_0}{8\pi D}\Big[r^2 + (r^2 + c^2)\ln c + c^2 \sum{}' \frac{\varrho^n}{n}\Big(\frac{1}{n-1} - \frac{\varrho^2}{n+1}\Big)\cos n\theta\Big] \qquad (r \leqslant c) \tag{17a}$$

$$= -\frac{P_0}{8\pi D}\left[c^2 + (r^2 + c^2)\ln r + c^2 \sum{}' \frac{\varrho^{-n}}{n}\left(\frac{\varrho^2}{n-1} - \right.\right.$$

$$\left.\left. - \frac{1}{n+1}\right)\cos n\theta\right] \qquad (r \geqslant c), \tag{17b}$$

where the accent on $\sum$ indicates that the summation is taken only over integral multiples of m.

With the abbreviation $s + 2 = s'$ the particular solution W(r) of (2) corresponding to the load (4) may be taken as

$$W(r) = p_0 r^{s'}/(s^2 s'^2 D) \qquad \text{in case I} \qquad (s, s' \neq 0), \tag{18a}$$

$$W(r) = p_0 r^2 \ln^2 r/(8d) \qquad \text{in case II} \qquad (s = 0), \tag{18b}$$

$$W(r) = p_0 \ln^2 r/(8D) \qquad \text{in case III} \qquad (s' = 0). \tag{18c}$$

The deflection $w(r,\theta)$ at any point $N(r,\theta)$ of the circular annular plate will now be assumed in the form

$$\frac{D}{p_0} w(r,\theta) = \frac{D}{p_0} W(r) - \frac{k}{m} \sum_{\iota=1}^{m} R_\iota^2 \ln R_\iota + L_0(r) + \sum{}' L_n(r) \cos n\theta, \tag{19}$$

where
$$k = P_0/(8\pi p_0), \tag{20}$$

$$L_0(r) = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r, \tag{21a}$$

$$L_n(r) = A_n r^n + B_n r^{-n} + C_n r^{2+n} + D_n r^{2-n}, \tag{21b}$$

A_n, B_n, C_n and D_n ($n = 0, m, 2m, 3m, \ldots$) are real constants to be determined. The vanishing of the deflection at the m points of support N_ι gives

$$A_0 = kc^2(1 + 2\ln c + 2S_m) - \frac{D}{p_0} W(c) - B_0 c^2 - C_0 \ln c - D_0 c^2 \ln c - \sum{}' L_n(c), \tag{22}$$

where

$$S_m = \sum{}' \frac{1}{n(n^2-1)} = \frac{1}{2m} \sum_{\iota=1}^{m} (1 - \cos \iota\alpha)\ln[2(1 - \cos \iota\alpha)] - 1/2, \tag{23}$$

on using equations (1a), (2a), (3a) and (5b) of the Appendix. Hence

$$S_2 = \ln 2 - 1/2, \qquad S_3 = 1/2(\ln 3 - 1), \qquad S_4 = 1/4(3 \ln 2 - 2). \tag{24}$$

Substitution for A_0 from (22) in (19) gives

$$\begin{aligned}\frac{Dw}{p_0} = \frac{D}{p_0}\{W(r) - W(c)\} &+ kc^2\left\{1 + 2\ln c + 2S_m - \frac{1}{m}\sum_{t=1}^{m}\frac{R_t^2}{c^2}\ln R_t\right\} \\ &+ B_0(r^2 - c^2) + C_0\ln(r/c) + D_0(r^2\ln r - c^2\ln c) \\ &+ \sum{}'[A_n(r^n\cos n\theta - c^n) + B_n(r^{-n}\cos n\theta - c^{-n}) \\ &+ C_n(r^{2+n}\cos n\theta - c^{2+n}) + D_n(r^{2-n}\cos n\theta - c^{2-n})]. \end{aligned} \tag{25}$$

If w^j ($j = 1, 2$) are the deflections at points in regions 1 and 2, then equations (17) and (19) lead to

$$\begin{aligned}\frac{Dw^j}{p_0} &= \frac{D}{p_0}W(r) + A_0^j + B_0^j r^2 + C_0^j\ln r + D_0^j r^2\ln r \\ &+ \sum{}'(A_n^j r^n + B_n^j r^{-n} + C_n^j r^{2+n} + D_n^j r^{2-n})\cos n\theta \\ &= \frac{D}{p_0}\{W(r) - W(c)\} + B_0^j(r^2 - c^2) + C_0^j\ln\frac{r}{c} + D_0^j(r^2\ln r - c^2\ln c) \\ &+ \sum{}'[A_n^j(r^n\cos n\theta - c^n) + B_n^j(r^{-n}\cos n\theta - c^{-n}) + C_n^j(r^{2+n}\cos n\theta - \\ &- c^{2+n}) + D_n^j(r^{2-n}\cos n\theta - c^{2-n})], \end{aligned} \tag{26}$$

where

$$\left.\begin{aligned} A_0^1 &= A_0 - kc^2\ln c, & B_0^1 &= B_0 - k\ln ec, \\ C_0^1 &= C_0, & D_0^1 &= D_0, \\ A_n^1 &= A_n - \frac{kc^{2-n}}{n(n-1)}, & B_n^1 &= B_n, \\ C_n^1 &= C_n + \frac{kc^{-n}}{n(n+1)}, & D_n^1 &= D_n; \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} &A_0^2 = A_0 - kc^2, && B_0^2 = B_0, \\ &C_0^2 = C_0 - kc^2, && D_0^2 = D_0 - k, \\ &A_n^2 = A_n, && B_n^2 = B_n + \frac{kc^{2+n}}{n(n+1)} \\ &C_n^2 = C_n, && D_n^2 = D_n - \frac{kc^n}{n(n-1)}. \end{aligned}\right\} \quad (28)$$

There remains now the determination of the constants B_0, C_0, D_0, A_n, B_n, C_n and D_n ($n = m, 2m, 3m, \ldots$) from the boundary conditions along Γ_1 and Γ_2. Substituting from (26) in (7) and (8) and equating to zero the coefficients of $\cos n\theta$ ($n = 0, m, 2m, 3m, \ldots$) in the resulting identities we obtain the following sets of equations:

$$2(\nu + 1)B_0^j + \frac{\nu - 1}{c_j^2} C_0^j + \{\nu + 3 + 2(\nu + 1)\ln c_j\}D_0^j + \frac{D}{p_0}\{W''(c_j) + $$

$$+ \frac{\nu}{c_j} W'(c_j)\} = 0, \quad (29a)$$

$$\frac{4D_0^j}{c_j} + \frac{D}{p_0}\{W'''(c_j) + \frac{1}{c_j} W''(c_j) - \frac{1}{c_j^2} W'(c_j)\} = 0; \quad (29b)$$

$$(n-1)A_n^j c_j^{n-2} + (n+1)B_n^j c_j^{-n-2} + \left(1 + \frac{1}{n}\right)(n + \varkappa - 1)C_n^j c_j^n + \left(1 - \frac{1}{n}\right)(n -$$

$$- \varkappa + 1)D_n^j c_j^{-n} = 0, \quad (30a)$$

$$(n-1)A_n^j c_j^{n-2} - (n+1)B_n^j c_j^{-n-2} + \left(1 + \frac{1}{n}\right)(n - \varkappa - 1)C_n^j c_j^n - \left(1 - \frac{1}{n}\right)(n +$$

$$+ \varkappa + 1)D_n^j c_j^{-n} = 0, \quad (30b)$$

where

$$j = 1, 2, \qquad \varkappa = (3 + \nu)/(1 - \nu). \quad (31)$$

For convenience we also introduce the two symbols

$$\sigma = 1 - \nu, \qquad \beta = (1 - \nu)/(1 + \nu). \quad (32)$$

It is to be noted that the particular solutions (18) appear only in (29a, b) and do not enter into (30a, b). Equations (29a, b) must therefore be solved separately in the three cases of loading s,s′ ≠ 0, s = 0 and s′ = 0. Introducing (18) in (29b) yields

$$D_0^1 = -c_1^s/4s, \qquad D_0^2 = -c_2^s/4s \qquad \text{in case I}, \tag{33a}$$

$$D_0^1 = -\tfrac{1}{4}\ln ec_1, \qquad D_0^2 = -\tfrac{1}{4}\ln ec_2 \qquad \text{in case II}, \tag{33b}$$

$$D_0^1 = \tfrac{1}{8}\, c_1^{-2}, \qquad D_0^2 = \tfrac{1}{8}\, c_2^{-2} \qquad \text{in case III}. \tag{33c}$$

These values are consistent with the two equations $D_0^1 = D_0$ in (27) and $D_0^2 = D_0 - k$ in (28), where k is given by (20) and (12). Substituting in (29a) for W(c) from (18), for D_0^1, D_0^2 from (33), for B_0^j, C_0^j from (27), (28) and solving the resulting linear equations in B_0, C_0 we obtain the following values in the three cases of loading:

Case I. s,s′ ≠ 0, $\quad k = (c_2^s - c_1^s)/4s$.

$$\left.\begin{aligned} B_0 &= \frac{1}{4s(c_2^2 - c_1^2)}\Big[c_2^{s'}\ln c_2 - c_1^{s'}\ln c_1 + (c_1^s - c_2^s)(\tfrac{1}{2}\beta c^2 + c_1^2\ln ec) \\ &\quad + (c_1^{s'} - c_2^{s'})\Big(\frac{1}{s} - 1 - \frac{s\beta}{2s'}\Big)\Big], \\ C_0 &= \frac{1}{2s\beta(c_1^{-2} - c_2^{-2})}\Big[(c_1^s - c_2^s)\Big\{\frac{1}{s} + \tfrac{1}{2}\beta\Big(\lambda_2^2 - \frac{s}{s'}\Big)\Big\} - c_1^s\ln\lambda_1 - c_2^s\ln\lambda_2\Big]; \end{aligned}\right\} \tag{34a}$$

$$\begin{aligned} B_0^1 &= \frac{1}{4s(c_2^2 - c_1^2)}\Big[c_2^{s'}\ln c_2 - c_1^{s'}\ln c_1 + (c_1^s - c_2^s)(\tfrac{1}{2}\beta c^2 + c_1^2 + c_2^2\ln c) \\ &\quad + (c_1^{s'} - c_2^{s'})\Big(\frac{1}{s} - 1 - \frac{s\beta}{2s'}\Big)\Big], \quad C_0^1 = C_0; \end{aligned} \tag{34b}$$

$$\begin{aligned} B_0^2 = B_0, \quad C_0^2 &= \frac{1}{2s\beta(c_1^{-2} - c_2^{-2})}\Big[(c_1^s - c_2^s)\Big\{\frac{1}{s} + \tfrac{1}{2}\beta\Big(\lambda_1^{-2} - \frac{s}{s'}\Big)\Big\} - c_1^s\ln\lambda_1 - \\ &\quad - c_2^s\ln\lambda_2\Big], \end{aligned} \tag{34c}$$

where λ_1, λ_2 and λ are defined by (1).

Case II. s = 0, $k = -\tfrac{1}{4}\ln\lambda$.

$$\left.\begin{aligned} B_0 &= \frac{1}{8}\left[\tfrac{1}{2}(1+\beta) + \ln^2 ec_2 + \frac{\ln\lambda}{1-\lambda^2}\left(\beta\lambda_2^2 + \lambda^2 \ln\frac{\lambda_2}{\lambda_1}\right)\right], \\ C_0 &= \frac{c_1^2 \ln\lambda}{4(1-\lambda^2)}\left\{\lambda_2^2 + \beta^{-1}\ln\frac{\lambda_2}{\lambda_1}\right\}; \end{aligned}\right\} \tag{35a}$$

$$B_0^1 = \frac{1}{8}\left[\tfrac{1}{2}(1+\beta) + \ln^2 ec_1 + \frac{\ln\lambda}{1-\lambda^2}\left(\beta\lambda_2^2 + \ln\frac{\lambda_2}{\lambda_1}\right)\right], \quad C_0^1 = C_0. \tag{35b}$$

$$B_0^2 = B_0, \quad C_0^2 = \frac{c_1^2 \ln\lambda}{4(1-\lambda^2)}\left\{\lambda_1^{-2} + \beta^{-1}\ln\frac{\lambda_2}{\lambda_1}\right\}. \tag{35c}$$

Case III. $s' = 0, \quad k = \tfrac{1}{8}(c_1^{-2} - c_2^{-2})$.

$$\left.\begin{aligned} B_0 &= \tfrac{1}{8} c_2^{-2}\left[\frac{1-\beta}{1-\lambda^2}\ln\lambda - \ln ec - \tfrac{1}{2}\beta\lambda_1^{-2}\right], \\ C_0 &= \frac{1}{8\sigma}\left[\nu + 5 + \frac{4\nu}{1-\lambda^2}(\ln c_1 - \lambda^2 \ln c_2)\right] - \frac{\ln ec}{4\beta} - \tfrac{1}{8}\lambda_2^2; \end{aligned}\right\} \tag{36a}$$

$$B_0^1 = \tfrac{1}{8} c_2^{-2}\left[\frac{1-\beta}{1-\lambda^2}\ln\lambda - \lambda^{-2}\ln ec - \tfrac{1}{2}\beta\lambda_1^{-2}\right], \quad C_0^1 = C_0; \tag{36b}$$

$$B_0^2 = B_0, \quad C_0^2 = \frac{1}{8\sigma}\left[\nu + 5 + \frac{4\nu}{1-\lambda^2}(\ln c_1 - \lambda^2 \ln c_2)\right] - \frac{\ln ec}{4\beta} - \tfrac{1}{8}\lambda_1^{-2}. \tag{36c}$$

Substituting in (30a, b) for A_n^1, B_n^1, C_n^1, D_n^1 from (27), for A_n^2, B_n^2, C_n^2, D_n^2 from (28) and solving the four linear equations obtained in A_n, B_n, C_n and D_n we find, after extensive algebraic manipulation

$$\begin{aligned} A_n = \frac{k\lambda_2^n c_2^{2-n} J_n}{n^2(n-1)} &[\varkappa^3 t_{-n,1} + \varkappa^2 \lambda_2^{-2n}\{(n-1)t_{1,0} + n\lambda_2^2 t_{-1,n}\} \\ &+ (n-1)\varkappa\{(n+1)t_{-n,1} + n\lambda_2^2 t_{0,-n}\} + (n^2-1)\lambda_2^{-2n} t_{0,1}(1 - n + n\lambda_1^{-2})], \end{aligned} \tag{37a}$$

$$\begin{aligned} B_n = \frac{k\lambda_2^{-n} c_2^{2+n} J_n}{n^2(n+1)} &[\varkappa^3 t_{1,n} + \varkappa^2 \lambda_2^{2n}\{(n+1)t_{1,0} + n\lambda_2^2 t_{n,1}\} \\ &+ (n+1)\varkappa\{(n-1)t_{1,n} + n\lambda_2^2 t_{n,0}\} + (n^2-1)\lambda_2^{2n} t_{1,0}(1 + n - n\lambda_2^2)], \end{aligned} \tag{37b}$$

$$C_n = \frac{kc^{-n}J_n}{n(n+1)}[\varkappa^2 t_{n,1} + \varkappa\lambda_2^{2n}\{(n+1)t_{0,-n} + n\lambda_2^2 t_{-n,-1}\}$$

$$+ (n+1)t_{1,0}(1 - n + n\lambda_1^{-2})], \tag{37c}$$

$$D_n = \frac{kc^{n}J_n}{n(n-1)}[\varkappa^2 t_{-1,n} + \varkappa\lambda_2^{-2n}\{(n-1)t_{n,1} + n\lambda_2^2 t_{-1,n}\}$$

$$+ (n-1)t_{-1,0}(1 + n - n\lambda_2^2)]; \tag{37d}$$

$$A_n^1 = \frac{k\lambda_2^n c_2^{2-n}J_n}{n^2(n-1)}[\varkappa^3 t_{-n,1} + \varkappa^2\lambda_2^{-2n}\{(n-1)t_{1,0} + n\lambda_2^2 t_{-n,1}\}$$

$$+ (n-1)\varkappa\{(n+1)t_{-n,1} + n\lambda_2^2 t_{0,-n}\} + (n^2-1)\lambda_2^{-2n}t_{0,1}(1 - n + n\lambda_2^2)], \tag{38a}$$

$$B_n^1 = B_n, \quad C_n^1 = \frac{kc^{-n}J_n}{n(n+1)}[\varkappa^2 t_{-1,-n} + \varkappa\lambda_2^{2n}\{(n+1)t_{0,-n} + n\lambda_2^2 t_{-n,-1}\}$$

$$+ (n+1)t_{0,-1}(1 - n + n\lambda_2^2)], \quad D_n^1 = D_n; \tag{38b}$$

$$A_n^2 = A_n, \quad B_n^2 = \frac{k\lambda_2^{-n}c_2^{2+n}J_n}{n^2(n+1)}[\varkappa^3 t_{1,n} + \varkappa^2\lambda_2^{2n}\{(n+1)t_{1,0} + n\lambda_2^2 t_{-1,-n}\}$$

$$+ (n+1)\varkappa\{(n-1)t_{1,n} + n\lambda_2^2 t_{n,0}\} + (n^2-1)\lambda_2^{2n}t_{1,0}(1 + n - n\lambda_1^{-2})],$$

$$C_n^2 = C_n, \tag{39a}$$

$$D_n^2 = \frac{kc^{n}J_n}{n(n-1)}[\varkappa^2 t_{-n,1} + \varkappa\lambda_2^{-2n}\{(n-1)t_{n,0} + n\lambda_2^2 t_{-1,n}\}$$

$$+ (n-1)t_{0,1}(1 + n - n\lambda_1^{-2})], \tag{39b}$$

where $n = m, 2m, 3m, \ldots$ and

$$J_n^{-1} = (n^2 + \varkappa^2 - 1)(\lambda^{-1} - \lambda)^2 - \varkappa^2(\lambda^{-n} - \lambda^n)^2, \quad t_{i,j} = \lambda^{2i} - \lambda^{2j}. \tag{40}$$

The constants in (25) and (26) are now completely determined and explicit formulae for the deflection surface are now given either by the single form (25) which is valid at any point of the circular annular plate ($c_2 \geqslant r \geqslant c_1$) or by the two forms (26) which are valid at points of regions 1 ($c \geqslant r \geqslant c_1$) and 2 ($c_2 \geqslant r \geqslant c$). Having found the deflection corresponding to the load (4) the principle of superposition can be applied to obtain the deflection surface of the annulus when it is subject to any symmetric loading that can be expanded in the form

$$p(r) = \sum_{-\infty}^{\infty} a_n r^n.$$

Formulae for the moments and shears at any point can be found by applying equations (5).

4. Symmetrically Loaded Circular Plate Supported by Point Forces Regularly Distributed on a Concentric Circle

Assuming that the load intensity is given by (4) where $s > 0$ and letting c_1 tend to zero in the foregoing values corresponding to case I we get

$$B_0 = k\left\{\ln ec_2 - \frac{1}{s} + \tfrac{1}{2}\beta\left(\frac{s}{s'} - \lambda_2^2\right)\right\}, \quad k = \frac{c_2^s}{4s}, \tag{41a}$$

$$C_0 = D_0 = B_n = D_n = 0, \tag{41b}$$

$$A_n = \frac{k}{\varkappa}\lambda_2^n c_2^{2-n}\left[\frac{1-\varkappa^2}{n^2(n-1)} - \frac{1}{n-1} + \frac{\lambda_2^2}{n}\right], \tag{41c}$$

$$C_n = \frac{k}{\varkappa}\lambda_2^n c_2^{-n}\left[\frac{1}{n} - \frac{\lambda_2^2}{n+1}\right]. \tag{41d}$$

Inserting these values in (25) leads to

$$\begin{aligned}\frac{Dw}{p_0} = {} & \frac{r^{s'} - c^{s'}}{s^2 s'^2} + k(r^2 - c^2)\left\{\ln ec_2 - \frac{1}{s} + \tfrac{1}{2}\beta\left(\frac{s}{s'} - \lambda_2^2\right)\right\} \\ & + kc^2\left\{1 + 2\ln c + 2S_m - \frac{1}{m}\sum_{t=1}^{m}\frac{R_t^2}{c^2}\ln R_t\right\} \\ & + \frac{k}{\varkappa}[(1-\varkappa^2)c_2^2\{T_m(u,\theta) - T_m(\lambda_2^2,0)\} + c_2^2\{I_m(u,\theta) - \\ & - I_m(\lambda_2^2,0)\} + r^2G_m(u,\theta) - c^2G_m(\lambda_2^2,0)],\end{aligned} \tag{42}$$

where

$$k = (c_2^s - c_1^s)/4s, \quad u = \varrho\lambda_2^2 = r\lambda_2/c_2, \tag{43a}$$

$$T_m(u,\theta) = \sum{}' \frac{u^n \cos n\theta}{n^2(n-1)}, \tag{43b}$$

$$I_m(u,\theta) = \sum{}' \left(\frac{\lambda_2^2}{n} - \frac{1}{n-1}\right) u^n \cos n\theta, \tag{43c}$$

$$G_m(u,\theta) = \sum{}' \left(\frac{1}{n} - \frac{\lambda_2^2}{n+1}\right) u^n \cos n\theta. \tag{43d}$$

The infinite series (43b, c, d) can be summed by using equations (1), (2), (3), (4) and (5a) of the Appendix and we get

$$c_2^2 I_m(u,\theta) + r^2 G_m(u,\theta) = \lambda_2^2 r^2 - \frac{1}{2m} \sum_{t=1}^{m} (r^2 + c^2 - 2\,c\,r\cos\theta_t)\ln(1 + u^2 - \\ - 2u\cos\theta_t). \tag{44}$$

Substituting from (14), (23) and (44) in (42) and setting $s = 2$, $s' = 4$ it is found that the resulting expression corresponding to uniform loading agrees with equation (34), p. 438 of Williams and Brinson (1974) on noting the difference between the notations used. Agreement with the more general result given by equations (3.2a), (3.2c), (3.3) and (3.13) of Bassali (1957) is also checked.

5. Symmetrically Loaded Circular Annulus Supported at Points Regularly Distributed Along the Outer Edge

The values of the constants in (26) ($j = 1$) are here obtained by letting c tend to c_2, setting $\lambda_2 = 1$ and $\lambda_1 = \lambda$ in equations (34b), (35b), (36b), (38a) and (38b). Thus we get

Case I

$$\left.\begin{aligned} B_0^1 &= \frac{c_1^s}{4s}\left[\tfrac{1}{2}\beta + \ln ec_2 + \frac{1}{\lambda^{-2} - 1}\left\{\left(\frac{1}{s} + \frac{\beta}{s'}\right)(1 - \lambda^{-s'}) - \ln\lambda\right\}\right], \\ C_0^1 &= \frac{c_1^{s'}}{2s\beta(1 - \lambda^2)}\left[\left(\frac{1}{s} + \frac{\beta}{s'}\right)(1 - \lambda^{-s}) - \ln\lambda\right], \quad D_0^1 = -\frac{c_1^s}{s}. \end{aligned}\right\} \tag{45a}$$

Case II

$$\left.\begin{aligned} B_0^1 &= \frac{1}{8}\left[\tfrac{1}{2}(1+\beta) + \ln^2 ec_1 + \frac{\ln\lambda}{1-\lambda^2}(\beta - \ln\lambda)\right], \\ C_0^1 &= \frac{c_1^2 \ln\lambda}{4(1-\lambda^2)}\left(1 - \frac{\ln\lambda}{\beta}\right), \quad D_0^1 = -\tfrac{1}{4}\ln ec_1. \end{aligned}\right\} \tag{45b}$$

Case III

$$\left.\begin{aligned} B_0^1 &= \tfrac{1}{8} c_1^{-2}\left[\frac{(1-\beta)\lambda^2 \ln\lambda}{1-\lambda^2} - \tfrac{1}{2}\beta - \ln ec_2\right], \\ C_0^1 &= \frac{1}{4}\left[\frac{1}{\sigma}\left(1 + \frac{2\nu\ln\lambda}{1-\lambda^2}\right) - \ln c_2\right], \quad D_0^1 = \tfrac{1}{8} c_1^{-2}. \end{aligned}\right\} \tag{45c}$$

In the three cases of loading the formulae (38a, b) for A_n^1, B_n^1, C_n^1 and D_n^1 simplify to

$$A_n^1 = \frac{4kc_2^{2-n}J_n}{\sigma n^2(n-1)}\left[\varkappa^2 t_{-n,1} + (n-1)\varkappa t_{-n,0} + (n^2-1)t_{0,1}\right], \tag{46a}$$

$$B_n^1 = \frac{4kc_2^{2+n}J_n}{\sigma n^2(n+1)}\left[\varkappa^2 t_{1,n} + (n+1)\varkappa t_{n,1} + (n^2-1)t_{1,0}\right], \tag{46b}$$

$$C_n^1 = \frac{4kc_2^{-n}J_n}{\sigma n(n+1)}\left[\varkappa t_{-1,-n} + (n+1)t_{0,-1}\right], \tag{46c}$$

$$D_n^1 = \frac{4kc_2^{n}J_n}{\sigma n(n-1)}\left[\varkappa t_{-1,n} + (n-1)t_{-1,0}\right], \tag{46d}$$

where k takes the appropriate value in each case.

6. Symmetrically Loaded Circular Plate Supported at Points Regularly Distributed Along the Edge

Letting c_1 tend to zero in the results of the foregoing article corresponding to the loading $p = p_0 r^{s-2}$ $(s > 0)$ gives

$$\left.\begin{aligned} &B_0^1 = -k\left(\frac{1}{s} + \frac{\beta}{s'}\right), \quad C_0^1 = D_0^1 = 0, \quad k = \frac{c_2^s}{4s}, \\ &A_n^1 = -\frac{4kc_2^{2-n}(n + \varkappa - 1)}{\varkappa\sigma n^2(n-1)}, \quad B_n^1 = 0, \quad C_n^1 = \frac{4kc_2^{-n}}{\varkappa\sigma n(n+1)}, \quad D_n^1 = 0. \end{aligned}\right\} \tag{47}$$

When these values are introduced in (26) (j = 1) the following formula for the deflection surface is obtained:

$$\frac{Dw}{p_0} = \frac{D}{s^2 s'^2 p_0}(r^{s'} - c^{s'}) + \frac{c_2^s}{4s}\left(\frac{1}{s} + \frac{\beta}{s'}\right)(c_2^2 - r^2)$$
$$+ \frac{c_2^{s'}}{s\varkappa\sigma}\sum{}' \frac{1}{n}\left[\left(\frac{1-\varkappa}{n} + \frac{\varkappa}{n-1}\right)(1 - \varrho^n \cos n\theta) + \frac{\varrho^{n+2}\cos n\theta - 1}{n+1}\right], \tag{48}$$

where $\varrho = r/c_2$. The infinite series appearing in (48) can be summed on noting that

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}, \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

and applying the results in the Appendix.

In the particular case of uniform loading and two point supports at the ends of a diameter the expression (48) agrees with Nadai's result given by equation (g), p. 294 of Timoshenko and Woinowsky-Krieger (1959).

7. Symmetrically Loaded Circular Annulus Supported at Points Regularly Distributed Along the Inner Edge

Letting c tend to c_1, putting $\lambda_1 = 1$ and $\lambda_2 = \lambda$ in equations (34c), (35c), (36c), (39a) and (39b) we find the following values for the constants in equation (26) (j = 2):

Case I

$$\left.\begin{aligned} &B_0^2 = \frac{c_2^s}{4s}\left[\tfrac{1}{2}\beta + \ln ec_1 + \frac{1}{1-\lambda^2}\left\{\left(\frac{1}{s} + \frac{\beta}{s'}\right)(\lambda^{s'} - 1) - \ln\lambda\right\}\right], \\ &C_0^2 = \frac{c_2^{s'}}{2s\beta(\lambda^{-2} - 1)}\left[(\lambda^s - 1)\left(\frac{1}{s} + \frac{\beta}{s'}\right) - \ln\lambda\right], \quad D_0^2 = -\frac{c_2^s}{4s}. \end{aligned}\right\} \tag{49a}$$

Case II

$$\left.\begin{aligned}
B_0^2 &= \frac{1}{8}\left[\tfrac{1}{2}(1+\beta) + \ln^2 ec_2 + \frac{\lambda^2\ln\lambda}{1-\lambda^2}(\beta+\ln\lambda)\right],\\
C_0^2 &= \frac{c_1^2\ln\lambda}{4(1-\lambda^2)}\left(1+\frac{\ln\lambda}{\beta}\right), \quad D_0^2 = -\tfrac{1}{4}\ln ec_2.
\end{aligned}\right\} \qquad (49b)$$

$$\left.\begin{aligned}
B_0^2 &= \tfrac{1}{8}c_2^{-2}\left[\frac{1-\beta}{1-\lambda^2}\ln\lambda - \tfrac{1}{2}\beta - \ln ec_1\right],\\
C_0^2 &= \frac{1}{4}\left[\frac{1}{\sigma}\left(1+\frac{2\nu\lambda^2\ln\lambda}{1-\lambda^2}\right) - \ln c_1\right], \quad D_0^2 = \tfrac{1}{8}c_2^{-2}.
\end{aligned}\right\} \qquad (49c)$$

In the three cases of loading we have

$$A_n^2 = \frac{4kc_1^{2-n}J_n}{\sigma n^2(n-1)}\left[\varkappa^2 t_{-1,n} + (n-1)\varkappa t_{0,n} + (n^2-1)t_{-1,0}\right], \qquad (50a)$$

$$B_n^2 = \frac{4kc_1^{2+n}J_n}{\sigma n^2(n+1)}\left[\varkappa^2 t_{-n,-1} + (n+1)\varkappa t_{0,-n} + (n^2-1)t_{0,-1}\right], \qquad (50b)$$

$$C_n^2 = \frac{4kc_1^{-n}J_n}{\sigma n(n+1)}\left[\varkappa t_{n,1} + (n+1)t_{1,0}\right], \qquad (50c)$$

$$D_n^2 = \frac{4kc_1^{n}J_n}{\sigma n(n-1)}(\varkappa t_{-n,1} + (n-1)t_{0,1}]. \qquad (50d)$$

8. Symmetrically Loaded Circular Plate Supported by a Single Column at the Centre

The limiting values of the constants given by (49a) and (50) as c_1 tends to zero are

$$\left.\begin{aligned}
B_0^2 &= \frac{c_2^s}{4s}\left(1+\ln c_2 - \frac{1}{s} + \frac{s\beta}{2s'}\right), \quad C_0^2 = 0, \quad D_0^2 = -\frac{c_2^s}{4s} \quad (s \geqslant 2),\\
A_n^2 &= B_n^2 = C_n^2 = D_n^2 = 0.
\end{aligned}\right\} \qquad (51)$$

When these values are substituted in (26) (j = 2) we obtain the formula

$$\frac{Dw}{p_0} = \frac{r^{s'}}{s^2 s'^2} + \frac{c_2^s r^2}{4s}\left(1 - \frac{1}{s} + \frac{s\beta}{2s'} - \ln\frac{r}{c_2}\right), \tag{52}$$

for the deflection surface of a circular plate subject to the loading (4) ($s > 0$) and supported by a single column at the centre, the edge being free. The same formula is arrived at by letting c tend to zero in (42). The maximum deflection at the edge is

$$w_{max} = \frac{p_0 c_2^{s'}}{4s'D}\left(\frac{1}{s'} + {}^1\!/_2\,\beta\varkappa\right) \tag{53}$$

References

Bassali, W.A. (1957) The transverse flexure of thin plates supported at several points, *Proc. Camb. phil. Soc. math. phys. Sci.* **53**: 728-743.

Bassali, W.A. (1960) Bending of a singularly loaded thin circular annulus with free boundaries, *J. Mech. Phys. Solids* **8**: 123-140.

Bassali, W.A. and **Gorgui, M.A.** (1960a) Flexural problems of circular ring plates and sectorial plates. I., *Proc. Camb. phil. Soc. math. phys. Sci.* **56**: 75-95.

Bassali, W.A. and **Gorgui, M.A.** (1960b) Flexural problems of circular ring plates and sectorial plates. II., *Proc. Camb. phil. Soc. math. phys. Sci.* **56**: 414-424.

Chantaramungkorn, K., Karasudhi, P. and **Lee, S.L.** (1973) Eccentrically loaded circular plates on columns, *J. struc. Div. Am. Soc. civ. Engrs* **99**: ST 1, 234-240.

Heap, J.C. (1968) Bending of circular plates under a variable symmetrical load. Bending of circular plates under a uniform load on a concentric circle, *J. Engineering for Industry, Trans. Am. Soc. civ. Engrs Series* B **90** (2): 268-294.

Kirstein, A.F. and **Wooley, R.M.** (1966 and 1968) Deflection of thin circular elastic plates under symmetrically distributed loading. Engineering and Instrumentation, Sec. C., *J. Res. natn. Bur. Stand.*, **70 C**, 4, 227-244 (1966); **72 C**, 1, 21-26 (1968).

Timoshenko, S. and **Woinowsky-Krieger, S.** (1959) *Theory of Plates and Shells*, McGraw-Hill, New York.

Williams, R. and **Brinson, H.F.** (1974) circular plate on multipoint supports, *J. Franklin Inst.* **297**: 429-447.

(Received 17/10/1982;
in revised form 27/03/1984)

Appendix

In this appendix we find the sums of the four infinite series

$$H_1 = \sum{}' \frac{x^n \cos n\theta}{n}, \quad H_2 = \sum{}' \frac{x^n \cos n\theta}{n+1},$$

$$H_3 = \sum{}' \frac{x^n \cos n\theta}{n-1} \quad \text{and} \quad H_4 = \sum{}' \frac{x^n \cos n\theta}{n^2},$$

where $1 \geqslant x \geqslant 0$, $\pi \geqslant \theta \geqslant 0$ and the accent on $\sum$ indicates that $n = m, 2m, 3m, \ldots$ $(m \geqslant 2)$.

Let $\zeta_t = xe^{-i\theta_t}$ where $\theta_t = \theta - \alpha_t$,

$\alpha_t = (t-1)\alpha$ $(t = 1, 2, \ldots, m)$; $\alpha = 2\pi/m$.

(i) To obtain the value of H_1 we consider the function

$$f(\zeta_1,\zeta_2,\ldots,\zeta_m) = -\sum_{t=1}^{m} \ln(1-\zeta_t) = \sum_{t=1}^{m}\sum_{n=1}^{\infty}\frac{\zeta_t^n}{n}.$$

$$\operatorname{Re} f = \sum_{n=1}^{\infty}\frac{x^n}{n}\sum_{t=1}^{m}\cos n\theta_t = -\sum_{t=1}^{m}\operatorname{Re}\ln(1-\zeta_t).$$

Applying equation (16) we get

$$H_1 = -\frac{1}{2m}\sum_{t=1}^{m}\ln(1+x^2-2x\cos\theta_t). \tag{1a}$$

The same series can be written in the form

$$H_1 = \sum_{k=1}^{\infty}\frac{x^{mk}\cos mk\theta}{mk} = \frac{1}{m}\sum_{k=1}^{\infty}\frac{y^k\cos k\phi}{k} \quad \text{where } y = x^m, \quad \phi = m\theta.$$

Hence we have

$$H_1 = \frac{1}{m}\operatorname{Re}\sum_{k=1}^{\infty}\frac{(ye^{i\phi})^k}{k} = -\frac{1}{m}\operatorname{Re}\ln(1-ye^{i\phi}) =$$

$$-\frac{1}{2m}\ln(1+x^{2m}-2x^m\cos m\theta). \tag{1b}$$

(ii) To evaluate H_2 we consider the function

$$g(\zeta_1,\zeta_2,\ldots,\zeta_m) = \sum_{t=1}^{m}\sum_{n=1}^{\infty}\frac{\zeta_t^n}{n+1} = -\sum_{t=1}^{m}[1+\zeta_t^{-1}\ln(1-\zeta_t)].$$

$$\mathrm{Re}\, g = \sum_{n=1}^{\infty}\frac{x^n}{n+1}\sum_{t=1}^{n}\cos n\theta_t = mH_2 = -m - \frac{1}{x}\mathrm{Re}\, e^{i\theta_t}\ln(1-xe^{-i\theta_t}).$$

Hence

$$H_2 = -1 - \frac{1}{mx}\sum_{t=1}^{m}\left[\tfrac{1}{2}\cos\theta_t\ln(1+x^2-2x\cos\theta_t) - \sin\theta_t\tan^{-1}\frac{x\sin\theta_t}{1-x\cos\theta_t}\right]. \tag{2a}$$

Noting that

$$\frac{d}{dx}\{xH_2\} = \sum{}' x^n\cos n\theta = -1 + \sum_{k=0}^{\infty} y^k\cos k\phi = -1 + \mathrm{Re}\,\frac{1}{1-ye^{i\phi}},$$

we deduce that

$$H_2 = -\frac{1}{2} + \frac{1}{2x}\int_0^x \frac{1-t^{2m}}{1+t^{2m}-2t^m\cos m\theta}\,dt. \tag{2b}$$

(iii) The value of H_3 is found by considering the function

$$h(\zeta_1,\zeta_2,\ldots,\zeta_m) = \sum_{t=1}^{m}\sum_{n=2}^{\infty}\frac{\zeta_t^n}{n-1} = -\sum_{t=1}^{m}\zeta_t\ln(1-\zeta_t).$$

$$\mathrm{Re}\, h = \sum_{n=2}^{\infty}\frac{x^n}{n-1}\sum_{t=1}^{m}\cos n\theta_t = -x\,\mathrm{Re}\{e^{-i\theta_t}\ln(1-xe^{-i\theta_t})\}.$$

Thus

$$H_3 = -\frac{x}{m}\sum_{t=1}^{m}\left[\tfrac{1}{2}\cos\theta_t\ln(1+x^2-2x\cos\theta_t) + \sin\theta_t\tan^{-1}\frac{x\sin\theta_t}{1-x\cos\theta_t}\right]. \tag{3a}$$

Since $\frac{d}{dx}\left\{\frac{H_3}{x}\right\} = \frac{1}{x^2}\sum' x^n \cos n\theta = \frac{x^{m-2}\cos m\theta - x^{2m-2}}{1 + x^{2m} - 2x^m \cos m\theta}$,

we have

$$H_3 = x \int_0^x \frac{t^{m-2}(\cos m\theta - t^m)}{1 + t^{2m} - 2t^m \cos m\theta}\, dt. \tag{3b}$$

(iv) We now consider the function

$$F(\zeta_1,\zeta_2,\ldots,\zeta_m) = \sum_{t=1}^{m}\sum_{n=1}^{\infty}\frac{\zeta_t^n}{n^2} = -\sum_{t=1}^{m}\int_0^{\zeta_t}\frac{\ln(1-u)}{u}\,du.$$

$$\operatorname{Re} F = \sum_{n=1}^{\infty}\frac{x^n}{n^2}\sum_{t=1}^{m}\cos n\theta_t = -\sum_{t=1}^{m}\operatorname{Re}\int_0^1\frac{\ln(1-\zeta_t v)}{v}\,dv.$$

This leads to

$$H_4 = -\frac{1}{2m}\sum_{t=1}^{m}\int_0^1 \ln(1 + v^2x^2 - 2vx\cos\theta_t)\frac{dv}{v}\,. \tag{4a}$$

This series can also be written as

$$H_4 = \frac{1}{m^2}\sum_{k=1}^{\infty}\frac{y^k\cos k\phi}{k^2} = \frac{1}{m^2}\operatorname{Re}\sum_{k=1}^{\infty}\frac{(ye^{i\phi})^k}{k^2} = -\frac{1}{m^2}\operatorname{Re}\int_0^1\frac{\ln(1 - vye^{i\phi})}{v}\,dv$$

which yields

$$H_4 = -\frac{1}{2m^2}\int_0^1 \ln(1 + x^{2m}v^2 - 2x^m v\cos m\theta)\,\frac{dv}{v}\,, \tag{4b}$$

$$\sum{}'\frac{1}{n^2} = \frac{\pi^2}{6m^2}\,. \tag{4c}$$

Since

$$\frac{1}{n^2(n+1)} = \frac{1}{n^2} - \frac{1}{n} + \frac{1}{n+1}\quad,\quad \frac{1}{n^2(n-1)} = \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n^2}\,, \tag{5a}$$

$$\frac{2}{n(n^2-1)} = \frac{1}{n+1} + \frac{1}{n-1} - \frac{2}{n}, \tag{5b}$$

we can easily obtain the sums of the infinite series

$$\sum{}' \frac{x^n \cos n\theta}{n^2(n+1)}, \quad \sum{}' \frac{x^n \cos n\theta}{n^2(n-1)} \quad \text{and} \quad \sum{}' \frac{x^n \cos n\theta}{n(n^2-1)}.$$

سطح الانثناء لحلقة دائرية رقيقة محملة تحميلا عموديا متماثلا فوق عدد من الأعمدة

وديع عطا الله بسالى

قسم الرياضيات - كلية العلوم - جامعة الكويت - الكويت

فى هذا البحث أمكن إيجاد حل على شكل متسلسلة لا منتهية للإزاحة العمودية الصغيرة التى تحدث فى حلقة دائرية رقيقة واقعة تحت تأثير حمل عمودى متماثل موزع على اللوحة كلها عندما تكون اللوحة موضوعة فوق عدد من الأعمدة الموزعة بمسافات متساوية على محيط دائرة مركزية. وقد بحثت الحالات النهائية والحالات الخاصة للحل.