Arab Gulf J. scient. Res. **2** (2), pp. 593-605 (1984)

Generalized Open Spheres in Generalized Metric Spaces

Ali Ahmad Fora

Department of Mathematics, Yarmouk University, Irbid, Jordan

ABSTRACT. We shall define size functions and sizable spaces. These spaces are generalizations of metric spaces. We shall discuss some of their properties and then we discuss the metrizable space they induce under certain conditions on the sizable space. We shall also define open spheres and open balls and discuss the topologies they induce. Among the results we obtain, we have the following:

i) Every countably compact sizable space must be separable, and hence, it must have at most countably many discrete points.

ii) Every countably compact sizable space induces a compact Hausdorff metrizable space which is weaker than the original topology. We shall also discuss the metrizability of countably compact spaces as an application of our concepts.

1. Introduction

Let X be a nonempty set and let $d: X \times X \to [0,\infty)$ be a function satisfying the following conditions:

i) $d(x,y) = 0$ if and only if $x = y$.

ii) $d(x,y) = d(y,x)$ for all $x,y \in X$.

iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a metric function on X and the ordered pair (X,d) is called a metric space. The set of all open spheres in X forms a base for the metric topology on X which will be denoted by T(d). Several nice properties are carried by this metric space (X,T(d)). For example, every metric space is first countable, normal and Hausdorff. In metric spaces, separability is equivalent to second countability (Long 1971).

Let N,R denote the set of all natural numbers, the set of all real numbers, respectively. Let Cl A denote the closure of the set A. Let T($\mathcal{B}$) denote the topology generated by the base $\mathcal{B}$.

If (X,d) is any metric space and $A \subseteq X$, then $p \in \text{Cl } A$ if and only if $\inf\{d(p,a): a \in A\} = 0$. We also know that a countably compact metric space must have an ϵ-net for every $\epsilon > 0$. From this fact and because of the structure of the base of (X,T(d)), we conclude that every countably compact metric space must be separable. We can also prove for the same space that it is second countable and hence Lindelöf.

The familiar Sorgenfrey Line S is defined to be the space of real numbers with the class of all half open intervals [a,b), $a < b$, as a base. It is a well-known fact that S is hereditary Lindelöf, first countable, separable and nonmetrizable. It is also known that the Sorgenfrey plane $S \times S$ is not normal (Steen and Seebach 1970).

We shall discuss all of the above properties for generalized metric spaces. These spaces are called 'sizable spaces'.

2. Sizable Spaces

Let (X,T) be a topological space and let $\mathcal{B}$ be a base for T. Let $L: X \times X \cup \mathcal{B} \to [0,\infty)$ be a function. Then L is called a size function for (X,T($\mathcal{B}$)) if it satisfies the following conditions:

L1) $L(x,y) = 0$ if and only if $x = y$.

L2) $L(x,y) = L(y,x)$ for all $x,y \in X$.

L3) For any $x \in X$, for any open set U_x containing x, and for any positive real number r, there exists a basic open set $V_{x,r} \in \mathcal{B}$ such that $x \in V_{x,r} \subseteq U_x$ and $L(V_{x,r}) < r$.

L4) For any $V,V' \in \mathcal{B}$ and for any $x,x' \in V$; $y,y' \in V'$, we have

$$L(x',y') \leq L(x,y) + L(V) + L(V').$$

A space (X,T) is called a sizable space if the topology T on X has a base $\mathcal{B}$ and an associated size function L. This space will be denoted by (X,T($\mathcal{B}$),L).

For a nonempty set $A \subseteq X$, if $\{L(x,y): x,y \in A\}$ is a bounded set in R, then we define the L-diameter of A; denoted by $\delta_L(A)$; by

$$\delta_L(A) = \text{L-diameter of } A = \sup\{L(x,y): x,y \in A\}.$$

We shall write $\delta(A)$ for $\delta_L(A)$ if there will be no occasion for confusion. A size function L is called bounded if $\delta(X) < \infty$. If L is bounded, then we call $(X,T(\mathcal{B}),L)$ a bounded sizable space. The L-open sphere in $(X,T(\mathcal{B}),L)$ with center at p and radius $r > 0$ is denoted by $S_L(p,r)$ and is defined by

$$S_L(p,r) = \{x \in X: L(p,x) < r\}.$$

We shall write $S(p,r)$ for $S_L(p,r)$ if there will be no occasion for confusion. The L-open ball in $(X,T(\mathcal{B}),L)$ with center p and radius $r > 0$ is denoted by $B_L(p,r)$ and is defined by

$$B_L(p,r) = \cup \{V: p \in V, V \in \mathcal{B} \text{ and } L(V) < r\}.$$

We shall write $B(p,r)$ for $B_L(p,r)$ if there will be no occasion for confusion. It is clear that every L-open ball is open in $(X,T(\mathcal{B}))$. However, we are going to prove that every L-open sphere is open in $(X,T(\mathcal{B}))$. It is also important to notice that these open spheres (balls) are not necessarily basic open sets in $(X,T(\mathcal{B}))$ (see Example 2.2 below).

The obvious difference between size functions and metric functions is the triangle inequality. This, of course, will yield a major difference between sizable spaces and metrizable spaces. We shall explain ourselves through the following theorems and the following examples in this paper.

2.1 Theorem

Every metric space $(X,T(d))$ is a sizable space.

Proof

Define the function L by $L(x,y) = d(x,y)$, $x,y \in X$; and also define $L(V) = \sup\{d(x,y): x,y \in V\}$, where V is an open sphere in (X,d). Then for any V,V' d-open spheres, and for any $x,x' \in V$; $y,y' \in V'$, we have (use the triangle inequality of d):

$$L(x',y') = d(x',y') \leq d(x',x) + d(x,y) + d(y,y') \leq L(V) + L(x,y) + L(V').$$

Therefore, L is a size function for $(X,T(d))$, *i.e.*, $(X,T(d),L)$ is a sizable space.

The following example shows that the converse of *Theorem 2.1* is in general not true.

2.2 Example

(i) The Sorgenfrey Line S is a sizable space.

(ii) The Sorgenfrey plane $S \times S$ is a sizable space.

Proof

(i) Let $\mathcal{B} = \{[a,b): a < b\}$ and let L be defined by $L(x,y) = |x - y|$; $x,y \in S$ and $L([a,b)) = 2(b - a)$. Then, L is a size function for S.

(ii) Let $\mathcal{B} = \{[a,b) \times [c,d): a < b, c < d\}$ and let L be defined by $L[a,b), [c,d)) = |a - c| + |b - d|$; $a, b, c, d \in R$ and $L([a,b) \times [c,d)) = (b - a) + (d - c)$. Then, L is a size function for $S \times S$.

Actually, one can prove the following theorem (Fora 1983a).

2.3 Theorem

Let $(X,T_1),(Y,T_2)$ be topological spaces and let T_p denote the product topology on $X \times Y$. Then $(X \times Y,T_p)$ is sizable if and only if (X,T_1), (Y,T_2) are sizable spaces.

Example 2.2 illustrates that a sizable Lindelöf separable first countable space need not be second countable. This is one of the main differences between sizable spaces and metrizable spaces. Notice also that, in Example 2.2, we have $S(0,r) = (-r,r)$ while $B(0,r) = (-\frac{1}{2} r, \frac{1}{2} r)$ $(r > 0)$ and these sets are not basic open sets in S. We also observe that a sizable space need not be normal (see Example 2.2(ii)).

We noticed above that every metric function is a size function. However, Example 2.2(i) shows that the converse is not true.

One can prove the following theorem which illustrates some properties of size functions and sizable spaces (Fora 1983a).

2.4 Theorem

Let $(X,T(\mathcal{B}))$ be a sizable space with the size function L. Then we have the following:

i) $(X,T(\mathcal{B}))$ is a Hausdorff space.

ii) $\delta(V) \leqslant L(V)$ for every $V \in \mathcal{B}$.

iii) $L(V) = 0$ if and only if V has exactly one element; $V \in \mathcal{B}$

iv) Every point in X is a G_δ-set in X.

v) If $p \in Cl\, F$ where $F \subseteq X$, then $\inf\{L(p,x): x \in F\} = 0$.

vi) The converse of (v) is in general not true.

Although every point in a sizable space must be a G_δ-set, the space itself need not be first countable. We illustrate this fact in the following example.

2.5 Example

A sizable space need not be first countable.

Proof

Let X = R × R. Define basic open sets in X as follows:

$$V_r((x,mx)) = \{(x',mx'): |x' - x| < r\},\ r < |x|,\ 0 < r \leqslant 1;\ \text{if } x \neq 0;$$

$$V((0,0)) = \bigcup_{m \in R} \{(x',mx'): |x'| < r_m\} \cup \{(0,t): |t| < r\};\ \text{where } r, r_m \in (0,1]$$

$$V_r((0,y)) = \{(0,y'): |y - y'| < r\},\ r < |y|,\ 0 < r \leqslant 1;\ \text{if } y \neq 0.$$

This space is called the Radial Interval Topology (Steen and Seebach 1970). Define L as follows:

$$L((x,y),(z,t)) = ((x - z)^2 + (y - t)^2)^{1/2};\ x,y,z,t \in R;$$

$$L(V) = \sup\{L(x,y): x,y \in V\},\ \text{where } V \text{ is a basic open set.}$$

Then $(X,T(\mathcal{B}))$ is a sizable space and not first countable because it is not first countable at (0,0).

The following theorem characterizes the class of all size functions on a given topological space (X,T). For the proof, consult (Fora 1983a).

2.6 Theorem

(i) Let $(X,T(\mathcal{B}),L)$ be a sizable space. Let $V \in \mathcal{B}$ be a basic open set. Then for any $a,b \in V$ and for any $x \in X$, we have

$$L(x,b) \leqslant L(x,a) + L(V).$$

(ii) Let $(X,T(\mathcal{B}))$ be a topological space and let $L: X \times X \cup \mathcal{B} \to [0,\infty)$ be a function satisfying (L1), (L2), (L3) and the following condition:

L4)′ For any $V \in \mathcal{B}$ and for any $a,b \in V$, $x \in X$, we have

$$L(x,b) \leqslant L(x,a) + L(V).$$

Then L is a size function for $(X,T(\mathcal{B}))$.

3. Spheres In Sizable Spaces

We shall start this section by the following theorem.

3.1 Theorem

Let $(X,T(\mathcal{B}),L)$ be a sizable space and let $p \in X$, $r > 0$. Then we have the following:

i) $B(p,r)$ is always open in $(X,T(\mathcal{B}))$.

ii) $S(p,r)$ is always open in $(X,T(\mathcal{B}))$.

iii) $B(p,r) \subseteq S(p,r)$.

Proof

(i) It is clear that $B(p,r)$ is an open set in $(X,T(\mathcal{B}))$ because it is the union of open sets in $(X,T(\mathcal{B}))$.

(ii) To prove that $S(p,r)$ is open in $(X,T(\mathcal{B}))$, let $q \in S(p,r)$. Then $L(p,q) < r$, *i.e.*, $r - L(p,q) > 0$. By the use of (L3), there exists $V \in \mathcal{B}$ such that $q \in V$ and $L(V) < r - L(p,q)$. We claim that $V \subseteq S(p,r)$. To prove our claim, let $x \in V$. Then by *Theorem 2.6(i)*, we have

$$L(p,x) \leqslant L(p,q) + L(V) < L(p,q) + r - L(p,q) = r.$$

Therefore $x \in S(p,r)$; *i.e.*, our claim is completely verified. Hence, for each $q \in S(p,r)$, there exists $V \in \mathcal{B}$ such that $q \in V \subseteq S(p,r)$. Consequently, $S(p,r)$ is open in $(X,T(\mathcal{B}))$.

(iii) Let $x \in B(p,r)$. Then, there exists $V \in \mathcal{B}$ such that $x,p \in V$ and $L(V) < r$. By the use of *Theorem 2.4(ii)*, we get $L(p,x) \leqslant L(V) < r$. Consequently, $x \in S(p,r)$. Hence, the proof of the theorem is completed.

In what follows, let $\mathcal{B}^*$, $\hat{\mathcal{B}}$ be defined as follows:

$$\mathcal{B}^* = \{S(p,r)\colon p \in X,\ r > 0\},$$

$$\hat{\mathcal{B}} = \{B(p,r)\colon p \in X,\ r > 0\}.$$

Then, we have the following results.

3.2 Theorem

Let $(X,T(\mathcal{B}))$ be a sizable space with a size function L. Then, we have the following:

(i) If $\mathcal{B}^*$ is a base for some topology on X, then $(X,T(\mathcal{B}^*))$ is a T_1-space.

(ii) If $\hat{\mathcal{B}}$ is a base for some topology on X, then $(X,T(\hat{\mathcal{B}}))$ is a T_1-space.

(iii) If $\mathcal{B}^*,\hat{\mathcal{B}}$ are bases for some topologies on X, then

$$T(\mathcal{B}^*) \subseteq T(\hat{\mathcal{B}}) \subseteq T(\mathcal{B}).$$

Proof

(i) Let $p,q \in X$ such that $p \neq q$. Then $L(p,q) = r > 0$. Let $t = \frac{1}{2} r$. Then, it is clear that $q \in S(q,t)$, $p \notin S(q,t)$, $p \in S(p,t)$, $q \notin S(p,t)$.

(ii) Let $p,q \in X$ such that $p \neq q$. Then $L(p,q) = r > 0$. Let $t = \frac{1}{2} r$. Then it is clear that $q \in B(q,t)$, $p \notin B(q,t)$, $p \in B(p,t)$, $q \notin B(p,t)$ (use *Theorem 2.4(ii)*).

(iii) It is easy to prove (iii) by using *Theorem 3.1(i)* and *Theorem 3.1(iii)*.

Although $(X,T(\mathcal{B}^*))$ and $(X,T(\hat{\mathcal{B}}))$ are T_1-spaces, they need not be Hausdorff. The following example illustrates this fact.

3.3 Example

For sizable space $(X,T(\mathcal{B}),L)$, the space $(X,T(\mathcal{B}^*))$ is not necessarily a Hausdorff space.

Proof

Let $X = [0,1]$, $\mathcal{B} = \{\{0\}, \{1\}, (a,b): 0 < a < b < 1\}$. Define $L(x,y) = |x - y|$ if $xy \neq 0$, $L(x,0) = L(0,x) = x(1 - x)$ if $x \neq 1$, $L(0,1) = L(1,0) = 1$, $L(x,x) = 0$; $x,y \in X$, and we also define $L((a,b)) = b - a$, $0 < a < b < 1$, $L(\{0\}) = L(\{1\}) = 0$. Then $\mathcal{B}$ is a base for some topology on X, and moreover L is a size function for $(X,T(\mathcal{B}))$, *i.e.*, $(X,T(\mathcal{B}),L)$ is a sizable space. By doing some calculations, we find that

$$S(0,r) = [0,\tfrac{1}{2}(1 - t)) \cup (\tfrac{1}{2}(1 + t),1), \text{ where } t = (1 - 4r)^{1/2} \text{ and } 0 < r < \tfrac{1}{4},$$
$$S(1,r) = (1 - r,1], \; 0 < r < 1.$$

Therefore $S(0,r) \cap S(1,r') \neq \emptyset$ for every $r,r' > 0$. Hence $(X,T(\mathcal{B}^*))$ is not Hausdorff. Notice also that $(X,T(\mathcal{B}))$ is a second countable locally compact space and is the union of two metrizable subspaces which are both open and closed in X.

Although $(X,T(\mathcal{B}^*))$ need not be a Hausdorff space, it will be Hausdorff under certain conditions on $(X,T(\mathcal{B}))$ as we shall see in the next section.

4. Open Spheres In Countably Compact Sizable Spaces

Let $(X,T(\mathcal{B}),L)$ be a sizable space and ϵ a positive real number. The finite set $A \subseteq X$ is called an ϵ-net for X iff for each point $x \in X$ there exists a point $a \in A$ such that $L(x,a) < \epsilon$. Recall that a space X is countably compact iff every countable open cover of X has a finite subcover for X. The following theorem characterizes the class of all countably compact T_1-spaces (Long 1971).

4.1. Theorem

Let (X,T) be a T_1-space. Then X is countably compact if and only if every infinite subset of X has a cluster(accumulation) point in X.

The following is our first result in this section.

4.2 Theorem

Let $(X,T(\mathcal{B}))$ be a countably compact sizable space with a size function L, and let $A \subseteq X$. Then we have

$$p \in \text{Cl } A \text{ if and only if } \inf\{L(p,a): a \in A\} = 0.$$

Proof

Let $\inf\{L(p,a): a \in A\} = 0$. Then for each $n \in N$, there exists a point $a_n \in A$ such that $L(p,a_n) < 1/n$. We have two cases to consider:

Case I. If $\{a_1,a_2,\ldots\}$ is a finite set, then there exists $k \in N$ such that $a_k = a_i$ for infinitely many i's. Therefore $L(p,a_k) = L(p,a_i) < 1/i$ for infinitely many i's. Hence $L(p,a_k) = 0$, *i.e.*, $p = a_k$. Consequently $p \in \text{Cl } A$.

Case II. If $\{a_1,a_2,\ldots\}$ is an infinite set, then there exists a point $q \in X$ such that q is a cluster point of $\{a_1,a_2,\ldots\}$ (we have used *Theorem 4.1* and *Theorem 2.4(i)*). We claim that $p = q$. To prove our claim, suppose on the contrary that $p \neq q$. Then $L(p,q) = r > 0$. Take a fixed natural number $m \in N$ such that $1/m < r$. Applying (L3), there exist two basic open sets $V_p,V_q \in \mathcal{B}$ such that $p \in V_p$, $q \in V_q$ and $L(V_p) < 1/(4m)$, $L(V_q) < 1/(4m)$. Since q is a cluster point of $\{a_1,a_2,\ldots\}$, therefore there exists a natural number $k > 4m$ such that $a_k \in V_q$. Applying (L4), we get $L(p,q) \leq L(p,a_k) + L(V_p) + L(V_q) < 1/k + 2/(4m) < 1/m$. This implies that $r = L(p,q) < 1/m$ which is impossible. Hence, the proof of the theorem is completed because of *Theorem 2.4(v)*.

Using the above theorem, one can prove the following theorem (Fora 1983b).

4.3 Theorem

Let $(X,T(\mathcal{B}),L)$ be a countably compact sizable space. Then X has an ϵ-net for each $\epsilon > 0$.

4.4 Theorem

Every countably compact sizable space is separable; and hence must have at most countably many discrete points.

The following lemma is needed for the next theorem.

4.5 Lemma

Let $(X,T(\mathcal{B}),L)$ be a countably compact sizable space and let $p \in X$, $r > 0$. Then, for any $q \in S(p,r)$ there exists $n \in N$ such that $S(q,1/n) \subseteq S(p,r)$.

Proof

Suppose on the contrary, that is, for some $q \in S(p,r)$ and for all $n \in N$, we have $S(q,1/n) \nsubseteq S(p,r)$. Therefore, there exists $x_n \in S(q,1/n)$ and $x_n \notin S(p,r)$ $(n \in N)$. Consequently, $L(q,x_n) < 1/n$ and $x_n \in X - S(p,r)$. By the use of *Theorem 4.2*, we have $q \in Cl(X - S(p,r))$. Using *Theorem 3.1(ii)*, we get $q \in X - S(p,r)$, *i.e.*, $q \notin S(p,r)$ which is a contradiction. This completes the proof of the Lemma.

4.6 Theorem

Let $(X,T(\mathcal{B}),L)$ be a countably compact sizable space. Then, the set of all open spheres in X forms a base $\mathcal{B}^*$ for some topology on X.

Proof

Easy by using the above *Lemma 4.5*.

The following lemma concerns open balls in sizable spaces. This lemma is needed for the proof of the next theorem.

4.7 Lemma

Let $(X,T(\mathcal{B}),L)$ be a countably compact sizable space and let $p \in X$, $r > 0$. Then for each $q \in B(p,r)$ there exists $n \in N$ such that $B(q,1/n) \subseteq B(p,r)$.

Proof

Suppose on the contrary, that is, for some $q \in B(p,r)$ and for every $n \in N$, we have $B(q,1/n) \nsubseteq B(p,r)$. Therefore, there exists $x_n \in B(q,1/n)$ and $x_n \notin B(p,r)$ $(n \in N)$. Consequently, there exists $V_n \in \mathcal{B}$ such that $x_n,q \in V_n$ and $L(V_n) < 1/n$. By the use of *Theorem 2.4(ii)*, we have $L(x_n,q) \leqslant L(V_n) < 1/n$. Hence $\inf\{L(x,q): x \in X - B(p,r)\} = 0$. Using *Theorem 4.2*, we get $q \in Cl(X - B(p,r))$. But $X - B(p,r)$ is a closed set in $(X,T(\mathcal{B}))$ (by *Theorem 3.1(i)*). Therefore $q \in X - B(p,r)$ which is a contradiction. This completes the proof of the lemma.

4.8 Theorem

Let $(X,T(\mathcal{B}),L)$ be a countably compact sizable space. Then, the set of all open balls in X forms a base for some topology on X.

Proof

Easy by using the above *lemma 4.7.*

4.9 Theorem

Let $(X,T(\mathcal{B}),L)$ be a countably compact sizable space, Then $(X,T(\mathcal{B}^*))$ is a second countable space.

Proof

Since $(X,T(\mathcal{B}))$ is a countably compact sizable space, therefore; by *Theorem 4.4*; there exists a countable $T(\mathcal{B})$-dense subset $D \subseteq X$. Using *Theorem 3.2(iii)*, we deduce that D is $T(\mathcal{B}^*)$-dense in X, *i.e.*, $(X,T(\mathcal{B}^*))$ is a separable space. Let $\mathcal{B}_1$ be defined by

$$\mathcal{B}_1 = \{S(x,r)\colon x \in D;\ r \text{ is any positive rational number}\}.$$

Then $\mathcal{B}_1$ is obviously a countable set. To prove that $\mathcal{B}_1$ is a base for $T(\mathcal{B}^*)$, let $U \in T(\mathcal{B}^*)$ and $y \in U$. Then, there exists a positive rational number r such that $S(y,r) \subseteq U$. Since $S(y,r)$ is a $T(\mathcal{B})$-open set, therefore there exists $V \in \mathcal{B}$ such that $y \in V \subseteq S(y,\frac{1}{4}r)$ and $L(V) < \frac{1}{4}r$. Since D is $T(\mathcal{B})$-dense, therefore there exists $x \in V \cap D$. This implies that $x \in S(y,\frac{1}{4}r)$; *i.e.*, $L(x,y) < \frac{1}{4}r$. Consequently, we have $y \in S(x,\frac{1}{2}r)$. To prove $S(x,\frac{1}{2}r) \subseteq S(y,r)$, let $z \in S(x,\frac{1}{2}r)$. Then $L(x,z) < \frac{1}{2}r$. Applying *Theorem 2.6(i)*, we get $L(z,y) \leq L(z,x) + L(V) < \frac{1}{2}r + \frac{1}{4}r < r$. Hence $z \in S(y,r)$. Thus, we have found that $y \in S(x,\frac{1}{2}r) \subseteq S(y,r) \subseteq U$, $x \in D$, $\frac{1}{2}r$ is a positive rational number. Consequently $\mathcal{B}_1$ is a countable base for $(X,T(\mathcal{B}^*))$, which completes the proof of the theorem.

4.10 Theorem

Let $(X,T(\mathcal{B}),L)$ be a countably compact sizable space. Then $(X,T(\mathcal{B}^*))$ is compact and moreover $\delta(X) < \infty$ (*i.e.*, L is a bounded size function).

Proof

Since $(X,T(\mathcal{B}))$ is a countably compact sizable space and $T(\mathcal{B}^*) \subseteq T(\mathcal{B})$ (*Theorem 3.2(iii)*), so by *Theorem 3.2(i)* and *Theorem 4.1*, $(X,T(\mathcal{B}^*))$ is also countably compact. By the use of *Theorem 4.9*, $(X,T(\mathcal{B}^*))$ is a Lindelöf space. Hence $(X,T(\mathcal{B}^*))$ is a compact space. To prove the second part of the theorem, we notice that the $T(\mathcal{B}^*)$-open countable cover $\underset{\sim}{C} = \{S(p,n)\colon n \in N\}$ (p is a fixed point in

X) has a finite subcover; *i.e.*, there exists k ∈ N such that X = S(p,k). Hence, the result.

Now, we can state and prove one of our main results concerning the metrizability of countably compact spaces.

4.11 Theorem

Let (X,T(𝓑)) be a countably compact space. Then, (X,T(𝓑)) is metrizable if and only if it is sizable.

Proof

If (X,T(𝓑)) is metrizable, then it is sizable according to *Theorem 2.1*. Conversely, let (X,T(𝓑)) be a sizable space with a size function L. We are going to prove that 𝓑* is a base for T(𝓑), *i.e.*, 𝓑,𝓑* are equivalent bases for T(𝓑). In order to prove our claim, let V ∈ 𝓑 and let p ∈ V. We claim that there exists n ∈ N such that S(p,1/n) ⊆ V. To prove our claim, suppose, on the contrary, that is for each n ∈ N we have S(p,1/n) ⊈ V. Therefore, for each n ∈ N there exists x_n ∈ X such that L(p,x_n) < 1/n and x_n ∈ X − V. By the use of *Theorem 4.2*, since inf{L(p,x): x ∈ X − V} = 0, therefore p ∈ Cl(X − V) (the closure is being taken in (X,T(𝓑))). But V is open in (X,T(𝓑)). Hence p ∈ X − V which is a contradiction; and hence 𝓑* is a base for T(𝓑). Consequently, we have T(𝓑) ⊆ T(𝓑*). Combining this result together with *Theorem 3.2(iii)*, we obtain that T(𝓑) = T(𝓑*). Using *Theorem 2.4(i)*, *Theorem 4.9*, *Theorem 4.10* and the classical *theorem of Urysohn* (Willard 1970), we get the required result, that (X,T(𝓑)) is metrizable.

As a final remark in this section, we may observe from the following example that (X,T(𝓑),L) may be a countably compact sizable space and hence (X,T(𝓑)) is metrizable but L will not be a metric for X. We also notice that the condition 'countably compact' is needed heavily for the sizable space (X,T(𝓑)), for otherwise the resulting space (X,T(𝓑*)) will not be even Hausdorff as we observed in Example 3.3.

4.12 Example

There exists a countably compact sizable space (X,T(𝓑),L) in which L is not a metric function.

Proof

Let X = {1,2,3} and let 𝓑 = {{x}: x ∈ X} be the base for the discrete topology T(𝓑) on X. Define L(1,2) = 1, L(1,3) = 2, L(2,3) = 4, L(a,b) = L(b,a), L(a,a) = 0, L({a}) = 0; a,b ∈ X. Then L is a size function and not metric. Note that (X,T(𝓑),L) is a countably compact sizable space.

5. Significance Of The Main Result

As we noticed that all theorems concerning metric spaces use heavily the triangle inequality of the metric space. *Theorem 2.1* asserts that all theorems holding for sizable spaces must hold for metrizable spaces. Because of this fact, our theorems may be considered as generalizations of the well-known theorems for metric spaces. If we take a look in Long 1971, (page 260) we can see how strong our generalizations are. Actually, all the theorems for metric spaces use the properties of open spheres, especially the one which states that all open spheres in a metric space form a base for some topology. This property depends, indeed, heavily on the triangle inequality of the metric. Therefore, the generalization of the triangle rule brings a distinction between the balls and spheres approach.

References

Fora, A.A. (1983a) New generalizations of metric spaces, *Iraqi J. Sci.*, to appear.

Fora, A.A. (1983b) On some remarks concerning countably compact spaces, *Dirasat*, to appear.

Long, P.E. (1971) *An Introduction to General Topology*, Charles E. Merrill Publishing company, 281 p.

Steen, L.A. and **Seebach, J.A., Jr.** (1970) *Counterexamples in Topology*, Holt, Rinehart and Winston Inc., New York, 210 p.

Willard, S. (1970) *General Topology*, Addison-Wesley Publishing Company, Inc., Reading, Massachussetts, 369 p.

(Received 12/03/1983;
in revised form 09/01/1984)

تعميم الكرات المفتوحة في الفراغات المترية المعممة

علي أحمد فوره

قسم الرياضيات - جامعة اليرموك - إربد - المملكه الاردنية الهاشمية

سوف نعــرّف في هذا البحث الــدالــة الحجمية والفـراغ الحجمي. تعـد هذه الفـراغـات تعميًا للفـراغات المترية المعـروفة. سوف ندرس هذه الفراغات الحجمية وسندرس أيضاً الفراغات المترية التي تنتج عن هذه الفراغات الحجمية بإدخال بعض الشروط على الفراغات الحجمية. ومن بعض النتائج التي حصلنا عليها ما يلي:

١. كل فراغ حجمي ومـتراص معـدود X يجب أن يكون قابلاً للفصل، ولهذا فإن مجموعة العناصر المنعزلة في X يجب أن تكون معدودة.

٢. كل فراغ حجمي ومـتراص معـدود ينتج فراغاً متراصاً مترياً أضعف من الفراغ الأصلي. هذا وسوف نعالج متركية الفضاءات المتراصة المعدودة كتطبيقات على بحثنا هذا.