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On the Group of Units of Z[A$_4$]

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ABSTRACT. The group of units of the integral group ring $Z[A_4]$ has been determined by Allen, P.J. and Hobby, C. In this note we use a different method to give a description of it, namely we consider $Z[A_4]$ as a module over the ring of integer Z. Since $Z[A_4]$ has a finite rank which is $|A_4|$, therefore we construct a Z-basis for it and we use this basis to describe $Z[A_4]$ and characterize its group of units.

Let $Z[A_4]$ denote the integral group ring where A_4 is the alternating group of degree 4 and let $U(Z[A_4])$ denote its group of units and let $U^+(Z[A_4])$ be the subgroup of $U(Z[A_4])$ which have coefficient sum 1. Then Allen and Hobby have shown that

$$U^+(Z[A_4]) \simeq \{X;\ X \ \varepsilon\ SL(3, Z)\} \quad \text{such that:}$$

1) $X \equiv B^i \pmod 2$ for some $i = 0, 1, 2$

where

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

2) If $X = \begin{bmatrix} x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \\ x_{10} & x_{11} & x_{12} \end{bmatrix}$ and if

$$t_0 = x_4 + x_8 + x_{12},\ t_1 = x_5 + x_9 + x_{10} \quad \text{and} \quad t_2 = x_6 + x_7 + x_{11}$$

are the pseudo traces of the matrix X then two pseudo traces are congruent to 0 modulo 4.

In this article, by using a different approach, we characterize $U^+(Z[A_4])$ and the result can be stated as follows:

Theorem

Let $Z[A_4]$ denote the integral group ring of the alternating group of degree 4 over the ring of the integers and let $U(Z[A_4])$ be its group of units and $U^+(Z[A_4])$ be the subgroup of $U(Z[A_4])$ which has coefficient sum 1 then

$$U^+(Z[A_4]) \cong \{X\colon X \in SL(3, Z)\} \quad \text{such that}$$

1) $X \equiv B^i \pmod 2$ for some $i = 0, 1, 2,\quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

2) If $X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$ then either

(I) $x_7 \equiv (x_6 - x_2) \pmod 4$
$x_8 \equiv (x_3 + x_4) \pmod 4$
$x_9 \equiv (x_5 - x_1 + 1) \pmod 4$

or:

(II) $x_7 \equiv (x_6 - x_2 + 1) \pmod 4$
$x_8 \equiv (x_3 + x_4) \pmod 4$
$x_9 \equiv (x_5 - x_1) \pmod 4$

or:

(III) $x_7 \equiv (x_6 - x_2) \pmod 4$
$x_8 \equiv (x_3 + x_4 - 1) \pmod 4$
$x_9 \equiv (x_5 - x_1) \pmod 4$

The Proof

The group A_4 is generated by the elements a, b subject to the relations $a^2 = b^3 = (ab)^3 = e$, and we shall use the identifications: $a = (12)(34)$, $b = (123)$.

It is well known that

$$C[A_4] \cong C \oplus C \oplus C \oplus C_{3\times 3}$$

where C is the field of complex numbers and $C_{3\times 3}$ denote the ring of 3×3 matrices over C such that (see Hall 1976)

$$(12)(34) \to \left(1, 1, 1, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$(123) \to \left(1, \omega, \omega^2, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right), \quad \omega^3 = 1.$$

If we denote the above isomorphism by ψ then by restricting ψ to $Z[A_4]$ we shall describe $\psi(Z[A_4])$ in the direct product of full matrices rings over C. Also since the identity representation and the 3-dimensional representation of A_4 are integral, we shall restrict ourself further to determine $\psi(Z[A_4])$ as a subring of $Z \oplus C \oplus C \oplus Z_{3\times 3}$ by constructing a Z-basis for it.

In order to construct the required basis for $\psi(Z[A_4])$, it is useful to list the elements of A_4 in the following order (I, (12)(34), (13)(24), (14)(23), (123), (142), (134), (243), (132), (124), (143), (234)) and to write the images of the elements of A_4 in a matrix form N where each row in N represents $\psi(x)$, $x \in A_4$ as a row vector

[*e.g.*, we write $\psi(123) = (1, \omega, \omega^2, 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0)$].

$$N = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & \omega & \omega^2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & \omega & \omega^2 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
1 & \omega & \omega^2 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & \omega & \omega^2 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
1 & \omega^2 & \omega & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & \omega^2 & \omega & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\
1 & \omega^2 & \omega & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & \omega^2 & \omega & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}$$

Now by applying an elementary Z-row operations on the matrix N we get the following matrix

$$\bar{N} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & \omega - 1 & \omega^2 - 1 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\ 0 & -3 & -3 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

We claim that the rows of the matrix $\bar{N}$ represent the required Z-basis of $\psi(Z[A_4])$. To prove this claim we first write the explicit form for the preimage of the basis elements expressed as a Z-combinations of the group elements which will be given in the following table in which the columns are indexed by the elements of A_4 and each row gives the corresponding coefficient of an element of the basis $\{e_i: i = 1, 2, \ldots, 12\}$ where e_i is the i^{th} row of the matrix $\bar{N}$.

	I	(12)(34)	(13)(24)	(14)(23)	(123)	(142)	(134)	(243)	(132)	(124)	(143)	(234)
e_1	1											
e_2	−1				1							
e_3			−1	−1	1				1			
e_4	1	−1										
e_5					1			−1				
e_6									1	−1		
e_7									1			−1
e_8	1		−1									
e_9					1	−1						
e_{10}					1	−1	−1	1				
e_{11}									1	−1	1	−1
e_{12}	1	1	−1	−1								

To show that the set $\{e_i\}$ generates the image of $Z[A_4]$ as a Z-module, it would be enough to prove that the above matrix is non-singular, and it has an integral inverse. For this purpose, we write the required inverse in which the columns are indexed by the elements of the basis and each row gives the corresponding coefficient of the image of the group elements.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}
I	1											
(12)(34)	1			−1								
(13)(24)	1							−1				
(14)(23)	1			−1				1				1
(123)	1	1										
(142)	1	1							−1			
(134)	1	1			−1				1	−1		
(243)	1	1			−1							
(132)	1	−1	1	−1								−1
(124)	1	−1	1	−1		−1						−1
(143)	1	−1	1	−1		−1	−1				1	−1
(234)	1	−1	1	−1			−1					−1

We have seen that the rows of the matrix $\bar{N}$ represent the required Z-basis for $\psi(Z[A_4])$.

Therefore, we can write $\psi(Z[A_4])$ as follows

$$\psi(Z[A_4]) = \left\{ \sum_{i=1}^{12} a_i e_i,\ a_i \varepsilon Z \right\}$$

$$= \{(a_1,\ a_1 + a_2(\omega - 1) - 3a_3,\ a_1 + a_2(\omega^2 - 1) - 3a_3,\ X\}$$

where:

$$X = \begin{bmatrix} a_1 - a_2 + 2a_4 & a_2 + a_3 + 2a_5 & a_3 + 2a_6 \\ a_3 + 2a_7 & a_1 - a_2 + 2(a_4 + a_8) & a_2 + a_3 + 2(a_5 + a_9) \\ a_2 + a_3 + 2a_9 + 4a_{10} & a_3 + 2(a_6 + a_7) + 4a_{11} & a_1 - a_2 + 2(a_3 + a_8) + 4a_{12} \end{bmatrix}$$

$$,\ a_i \varepsilon Z,\ i = 1, 2, \ldots, 12$$

It is clear that $\psi(Z[A_4]) \subset Z \oplus Z[\omega] \oplus Z[\omega] \oplus Z_{3 \times 3}$. Also, it can be shown easily that the units in the ring $Z[\omega]$ are $\pm\, \omega^i$, $i = 0, 1, 2$. Therefore for the units in $\psi(Z[A_4])$ we have

$$a_1 = \pm 1$$

$$a_1 + a_2(\omega - 1) - 3a_3 = \pm\, \omega^i \quad (1)$$

$$a_1 + a_2(\omega^2 - 1) - 3a_3 = \pm\, \omega^j \quad (2)$$

$$\text{and} \quad \det X = \pm 1$$

where

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix},$$

$$\begin{array}{lll} x_1 = a_1 - a_2 + 2a_4, & x_2 = a_2 + a_3 + 2a_5, & x_3 = a_3 + 2a_6 \\ x_4 = a_3 + 2a_7, & x_5 = a_1 - a_2 + 2(a_4 + a_8), & x_6 = a_2 + a_3 + 2(a_5 + a_9) \\ x_7 = a_2 + a_3 + 2a_9 + 4a_{10}, & x_8 = a_3 + 2(a_6 + a_7) + 4a_{11}, & x_9 = a_1 - a_2 + 2(a_3 + a_8) + 4a_{12} \end{array}$$

$$, a_i \varepsilon Z, \quad i = 1, 2, \ldots, 12.$$

Now let ϕ denote the projection of the direct product $Z \oplus Z[\omega] \oplus Z[\omega] \oplus X_{3 \times 3}$ onto $Z_{3 \times 3}$, then

$$\phi(\psi(Z[A_4])) = X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} = Y.$$

The mapping $\phi\psi$ is a ring homomorphism from $Z[A_4]$ onto Y. Therefore, it induces a group homomorphism of the group of units $U(Z[A_4]$ of $Z[A_4]$ onto the group of units $U(Y)$ of Y. In fact, this group homomorphism is an isomorphism. So we have

$$U(Z[A_4]) \simeq \left\{ X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} : \det X = \pm 1 \right\}.$$

We recall now that if G is a group, then the group of normalized units $U^+(R(G))$ [that is the units which have coefficient sum 1] of the group ring $R[G]$ where R is commutative ring with identity is defined to be the kernel of the identity representation $\mu : R[G] \rightarrow R$
i.e.

$$U^+(R[G]) = \ker \mu = \{u \varepsilon U(R[G]): \mu(u) = 1\}$$

Clearly $U^+(R[G]) \lhd U(R[G])$ and $U(R[G]) = U^+(R[G]) \, X U(R)$ where $U(R)$ is the group of units of R. (Cohn and Livingstone 1965; Al-Sohebany 1979). Therefore, for the case $G = A_4$, $R = Z$ we have $U^+(Z[A_4])$ is a normal subgroup of index 2 in $U(Z[A_4])$ obtained by taking a_1 to be 1, in such case we find after a direct computation that the solutions for the equations (2), (3) taking in consideration the various values of i are

I $a_2 = 0,\quad a_3 = 0$
II $a = 1\quad a_3 = 0$
III $a_2 = -1\quad a_3 = 1.$

Now we consider the case I.

By substituting the values of a_2, a_3 in the matrix X we have

$$\begin{array}{lll} x_1 = 1 + 2a_4, & x_2 = 2a_5, & x_3 = 2a_6 \\ x_4 = 2a_7 & x_5 = 1 + 2(a_4 + a_8), & x_6 = 2(a_5 + a_9) \\ x_7 = 2a_9 + 4a_{10}, & x_8 = 2(a_6 + a_7) + 4a_{11}, & x_9 = 1 + 2a_8 + 4a_{12} \end{array}$$

which means that x_1, x_5, x_9 are odd and x_i are even for $i = 2, 3, 4, 6, 7, 8$. Therefore $X \equiv I \pmod 2$.

Moreover

$$x_7 = 2a_9 + 4a_{10} = x_6 - x_2 + 4a_{10}$$
$$\Rightarrow x_7 \equiv (x_6 - x_2) \pmod 4.$$

Also

$$x_8 = 2(a_6 + a_7) + 4z_{11} = x_3 + x_4 + 4a_{11}$$
$$\Rightarrow x_8 \equiv (x_3 + x_4) \pmod 4.$$

Finally

$$x_9 = 1 + 2a_8 + 4a_{12} = x_5 - x_1 + 1 + 4a_{12}$$
$$\Rightarrow x_9 \equiv (x_5 - x_1 + 1) \pmod 4.$$

A similar argument can be applied for the other two cases to show that

II: $X \equiv B \pmod 2$ where $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$x_7 \equiv (x_6 - x_2 + 1) \pmod 4$
$x_8 \equiv (x_3 + x_4) \pmod 4$
$x_9 \equiv (x_5 - x_1) \pmod 4$

and

III: $X \equiv B^2 \pmod 2$
$x_7 \equiv (x_6 - x_2) \pmod 4$
$x_8 \equiv (x_3 + x_4 - 1) \pmod 4$
$x_9 \equiv (x_5 - x_1) \pmod 4.$

Finally, it is very easy to show that det $X \equiv 1 \pmod 4$ for all cases, since det $X = \pm 1$. This implies that det $X = 1$. This completes the proof of theorem.

References

Allen, P.J. and **Hobby, C.** (1980) A characterization of units in $Z[A_4]$, *J. Algebra* **66**: 534-543.

Al-Sohebany, A.A., (1979) *On The Units of Certain Group Rings*, Ph. D. Thesis, University of Birmingham.

Cohn, J.A. and **Livingstone, D.** (1965) On the structure of group algebras, I, *Can. J. Math.* **17**: 583-593.

Hall, M. (1976) *The Theory of Groups*, Chelsea, New York; pp. 298-304.

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زمرة الوحدات للحلقة $Z[A_4]$

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لقد عين كل من P.J. Allen و C. Hobby زمرة الوحدات للحلقة $Z[A_4]$. وفي هذا البحث وباستخدام طريقة أخرى نعطي وصفا للحلقة $Z[A_4]$ حيث نعتبرها نموذجاً (Module) على حلقة الأعداد الصحيحة Z . ولما كان بُعد النموذج $Z[A_4]$ محدوداً ويساوى رتبة الزمرة A_4 ، لذا فإننا سننشىء له قاعدة على حلقة الأعداد الصحيحة Z ومن ثم وباستخدام هذه القاعدة يتم وصف $Z[A_4]$ و كنتيجة لذلك يتم تحديد زمرة الوحدات لهذه الحلقة.