Arab Gulf J. scient. Res. **3** (1), pp. 249-258 (1985)

On Some (2,3,16)-Groups of Degree N, $16 \leqslant N \leqslant 25$

S.A. Al-Salman

Mathematics Department, College of Science, King Saud University, Riyad, Saudi Arabia

ABSTRACT. In this paper we prove that S_N is a (2,3,16)-group for $16 \leqslant N \leqslant 25$, and that A_N is a (2,3,16)-group for $N \neq 23$ and $18 \leqslant N \leqslant 25$.

It is known (Wielandt 1968) that if G is a primitive group of degree $N = p + k$, where p is prime and $k \geqslant 3$, and has an element of degree and order p, then G is either the alternating group A_N or the symmetric group S_N.

Now let G be a finite (l,m,n)-group, that is, a finite group generated by two elements x and y of orders l and m whose product has order n. That is to say, we have

$$x^l = y^m = (xy)^n = 1$$

By defining $t = (xy)^{-1}$ we can write the above relation in the form

$$x^l = y^m = t^n = xyt = 1$$

In other words, G is a homomorphic image of the finite or infinite (l,m,n)-group G^* defined by the presentation

$$R^l = S^m = T^n = RST = 1$$

(Coxeter and Moser 1980).

When G is expressed as a permutation group which acts transitively on N symbols, let the permutations x,y,z have respectively x_i i-cycles $(1 \leqslant i < l)$, y_j j-cycles $(1 \leqslant j < m)$, z_k k-cycles $(1 \leqslant k < n)$. Then there exists an integer $g \geqslant 0$ (Singerman 1970) such that

$$2g - 2 + \sum_{i=1}^{l-1} x_i\left(1 - \frac{i}{l}\right) + \sum_{j=1}^{m-1} y_j\left(1 - \frac{j}{m}\right) + \sum_{k=1}^{n-1} z_k\left(1 - \frac{k}{n}\right) =$$

$$N\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right) \ldots \qquad \text{(I)}$$

Since we are interested in the (2,3,16)-groups, we put $l = 2$, $m = 3$ and $n = 16$. Then it follows easily that

$$x_i = 0,\ i \neq 1$$

$$y_j = 0,\ j \neq 1$$

$$z_k = 0,\ k \neq 1,2,4,8.$$

Therefore the formula (I) becomes

$$24x_1 + 32y_1 + 45z_1 + 42z_2 + 36z_4 + 24z_8 = 5N + 96(1 - g) \ldots \qquad \text{(II)}$$

Application of (II) shows that $g = 0$ or 1 for $16 \leqslant N \leqslant 19$ and $g = 0$, 1 or 2 for $20 \leqslant N \leqslant 38$.

For a fixed N, the solutions of (II) give the possible cycle-structures of the generators of the (2,3,16)-groups of degree N. We seek two generators x and y say, such that

(i) $x^2 = y^3 = (xy)^{16} = 1$
(ii) $\langle x, y\rangle = S_N$ or A_N

Proposition 1

The symmetric group S_N is a (2,3,16)-group for $16 \leqslant N \leqslant 25$.

Proof

The proof will be provided as a series of lemmas.

Lemma 1.1

S_{16} is a (2,3,16)-group (Al-Salman 1973).

Proof

Let H = $\langle x, y\rangle$, where:

$x = (0,15)\ (2,14)\ (4,13)\ (5,8)\ (9,12)\ \underline{1}\ \underline{3}\ \underline{6}\ \underline{7}\ \underline{10}\ \underline{11}$ $\quad : 2^5.1^6$
$y = (1,2,15)\ (3,4,14)\ (5,9,13)\ (6,7,8)\ (10,11,12)\ \underline{0}$ $\quad : 3^5.1$
$z = xy = (0,1,2,\ldots,15)$ $\quad : 16$
$\sigma = yz^4 = (0,4,2,3,8,10,15,5,13,9,1,6,11)\ (7,12,14)$ $\quad : 13.3$

H is primitive, since the four 4-cycles of z^4 do not form the blocks of imprimitivity. Indeed since $\sigma \varepsilon H$, $H = S_{16}$.

Lemma 1.2

S_{17} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (2,15)\ (4,14)\ (5,6)\ (7,13)\ (8,9)\ (10,12)\ (11,16)\ \underline{0}\ \underline{1}\ \underline{3}$: $2^7.1^3$
$y = (0,1,2)\ (3,4,15)\ (5,7,14)\ (8,10,13)\ (11,16,12)\ \underline{6}\ \underline{9}$: $3^5.1^2$
$z = xy = (0,1,2,\ldots\quad ,15)\ \underline{16}$: 16.1
$\sigma = yz^4 = (0,5,11,16)\ (1,6,10)\ (2,4,3,8,14,9,13,12,15,7)$: $10.4.3$

$H = S_{17}$, since H is primitive and $\sigma \varepsilon H$.

Lemma 1.3

S_{18} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (0,15)\ (1,2)\ (3,14)\ (4,5)\ (6,13)\ (7,9)\ (8,16)\ (10,12)\ (11,17)$: 2^9
$y = (1,3,15)\ (4,6,14)\ (7,10,13)\ (8,16,9)\ (11,17,12)\ \underline{0}\ \underline{2}\ \underline{5}$: $3^5.1^3$
$z = xy = (0,1,2,\ldots\quad ,15)\ \underline{16}\ \underline{17}$: 16.1^2
$\sigma = yz^3 = (0,3,2,5,8,16,12,14,7,13,10)\ (1,6)\ (4,9,11,17,15)$: $11.5.2$

Now z and σ^5 show that H is 2-transitive and since $\sigma \varepsilon H$, $H = S_{18}$.

Lemma 1.4

S_{19} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (0,15)\ (2,14)\ (3,5)\ (6,13)\ (7,9)\ (4,16)\ (8,17)\ (10,12)\ (11,18)\ \underline{1}$: $2^9.1$
$y = (1,2,15)\ (3,6,14)\ (4,16,5)\ (7,10,13)\ (8,17,9)\ (11,18,12)\ \underline{0}$: $3^6.1$
$z = xy = (0,1,2,\ldots\quad ,15)\ \underline{16}\ \underline{17}\ \underline{18}$: 16.1^3
$\sigma = yz^4 = (0,4,16,9,12,15,5,8,17,13,11,18)\ (1,6,2,3,10)\ (7,14)$: $12.5.2$.

Therefore $H = S_{19}$, since H is primitive and $\sigma \varepsilon H$.

Lemma 1.5

S_{20} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (0,15)\ (1,2)\ (3,14)\ (5,13)\ (6,17)\ (7,11)\ (8,19)\ (10,18)\ (12,16)\ \underline{4}\ \underline{9} \quad : 2^9.1^2$
$y = (1,3,15)\ (4,5,14)\ (6,16,13)\ (7,12,17)\ (8,18,11)\ (9,10,19)\ \underline{0}\ \underline{2} \quad : 3^6.1^2$
$z = xy = (0,1,2,\ldots\quad ,15)\ (16,17)\ (18,19) \quad : 16.2^2$
$\sigma = yz^3 = (0,3,2,5,1,6,17,10,18,14,7,15,4,8,19,12,16)\ (9,13)\ \underline{11} \quad : 17.2.1$

Now σ and y^{z^9} show that H is 2-transitive, hence it is primitive. So $H = S_{20}$, since $\sigma \varepsilon H$.

Lemma 1.6

S_{21} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (2,15)\ (4,14)\ (5,17)\ (6,12)\ (7,19)\ (8,10)\ (9,20)\ (11,18)\ (13,16)\ \underline{0}\ \underline{1}\ \underline{3} \quad : 2^9.1^3$
$y = (0,1,2)\ (3,4,15)\ (5,16,14)\ (6,13,17)\ (7,18,12)\ (8,11,19)\ (9,20,10) \quad : 3^7$
$z = xy = (0,1,2,\ldots\quad ,15)\ (16,17)\ (18,19)\ \underline{20} \quad : 16.2^2.1$
$\sigma = yz^3 = (0,4,2,3,7,18,15,6)\ (1,5,16)\ (8,14)\ (9,20,13,17)\ (10,12)\ (11,19) \quad : 8.4.3.2.^3$

x^z *and* σ^3 indicate that H is 2-transitive. Hence $H = S_{21}$, since $\sigma \varepsilon H$.

Lemma 1.7

S_{22} is a (2,3,16)-group

Proof

Let $H = \langle x, y \rangle$, where:

$x = (0,15)\ (1,7)\ (2,17)\ (3,5)\ (4,20)\ (6,16)\ (8,14)\ (9,19)\ (10,12)\ (11,21)\ (13,18) \quad : 2^{11}$
$y = (1,8,15)\ (2,16,7)\ (3,6,17)\ (4,20,5)\ (9,18,14)\ (10,13,19)\ (11,21,12)\ \underline{0} \quad : 3^7.1$
$z = xy = (0,1,2,\ldots\quad ,15)\ (16,17)\ (18,19)\ \underline{20}\ \underline{21} \quad : 16.2^2.1^2$
$\sigma = yz^4 = (0,4,20,9,18,2,16,11,21)\ (1,12,15,5,8,3,10)\ (6,17,7)\ (13,19,14) \quad : 9.7.3^2$

Indeed the cycle-structures of the generators show that H must be primitive. Hence, $H = S_{22}$, since $\sigma\ \varepsilon\ H$.

Lemma 1.8

S_{23} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

x = (1,2) (3,15) (4,5) (6,14) (7,19) (8,21) (9,17) (10,12) (11,22) (13,16) (18,20) $\underline{0}$: $2^{11}.1$
y = (0,1,3) (4,6,15) (7,16,14) (8,20,19) (11,22,12) (10,13,17) (9,18,21) $\underline{2}$ $\underline{5}$: $3^7.1^2$
$z = xy$ = (0,1,2,... ,15) (16,17,18,19) (20,21) $\underline{22}$: 16.4.2.1
$\sigma = yz^3$ = (0,4,9,17,13,16,1,6,2,5,8,21,12,14,10) (7,19,11,22,15) (18,20) $\underline{3}$: 15.5.2.1

$H = S_{23}$, since H is primitive and $\sigma \varepsilon H$.

Lemma 1.9

S_{24} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

x = (2,15) (4,14) (6,13) (8,12) (9,19) (10,20) (11,16) (17,23) (18,21) $\underline{0}$ $\underline{1}$ $\underline{3}$ $\underline{5}$ $\underline{7}$ $\underline{22}$: $2^9.1^6$
y = (0,1,2) (3,4,15) (5,6,14) (7,8,13) (9,16,12) (10,21,19) (11,17,20) (18,22,23) : 3^8
$z = xy$ = (0,1,2,... ,15) (16,17,18,19) (20,21,22,23) : 16.4^2
$\sigma = yz^2$ = (0,3,6) (1,4) (5,8,15) (7,10,23,16,14) (9,18,20,13) (11,19,12) (17,22,21) $\underline{2}$: $5.4.3^4.2.1$

Now σ and x^2 show that H is 2-transitive. It then follows that $H = S_{24}$, since $\sigma \varepsilon H$.

Lemma 1.10

S_{25} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

x = (2,15) (3,19) (4,17) (5,13) (6,8) (7,24) (9,12) (10,23) (11,20) (14,16) (21,22) $\underline{0}$ $\underline{1}$ $\underline{18}$: $2^{11}.1^3$
y = (0,1,2) (3,16,15) (4,18,19) (5,14,17) (6,9,13) (7,24,8) (10,20,12) (11,21,23) $\underline{22}$: $3^8.1$

$z = xy = (0,1,2,\ldots,15)\,(16,17,18,19)\,(20,21,22,23)\,\underline{24}$ $\quad : 16.4^2.1$
$\sigma = yz^2 = (0,3,18,17,7,24,10,22,20,14,19,6,11,23,13,8,9,15,5)\,(1,4,16)\,\underline{2}\,\underline{12}\,\underline{21}$ $\quad : 19.3.1^3$

H is primitive, since it has no blocks of length 5. Hence, $H = S_{25}$ since $\sigma \in H$.

Poposition 2

The alternating group A_N is a (2,3,16)-group for $N \neq 23$ and $18 \leqslant N \leqslant 25$.

Proof

The proof will be provided as a series of lemmas.

Lemma 2.1

A_{18} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (2,15)\,(3,6)\,(4,10)\,(5,11)\,(7,14)\,(8,17)\,(9,12)\,(13,16)\,\underline{0}\,\underline{1}$ $\quad : 2^8.1^2$
$y = (0,1,2)\,(3,7,15)\,(4,11,6)\,(8,16,14)\,(9,13,17)\,(10,5,12)$ $\quad : 3^6$
$z = xy = (0,1,2,\ldots,15)\,(16,17)$ $\quad : 16.2$
$\sigma = yz^3 = (0,4,14,11,9)\,(1,5,15,6,7,2,3,10,8,17,12,13,16)$ $\quad : 13.5$

x and σ^5 show that H is 2-transitive. Hence $H = A_{18}$, since $\sigma \in H$.

Lemma 2.2

A_{19} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (0,15)\,(2,14)\,(4,13)\,(6,12)\,(7,17)\,(8,10)\,(9,18)\,(11,16)\,\underline{1}\,\underline{3}\,\underline{5}$ $\quad : 2^8.1^3$
$y = (1,2,15)\,(3,4,14)\,(5,6,13)\,(7,16,12)\,(8,11,17)\,(9,18,10)\,\underline{0}$ $\quad : 3^6.1$
$z = xy = (0,1,2,\ldots,15)\,(16,17)\,\underline{18}$ $\quad : 16.2.1$
$\sigma = yz^4 = (0,4,2,3,8,15,5,10,13,9,18,14,7,16)\,(1,6)\,(11,17,12)$ $\quad : 14.3.2$

Therefore $H = A_{19}$, since H is primitive and contains σ.

Lemma 2.3

A_{20} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$$x = (0,15)\ (1,2)\ (3,14)\ (4,6),\ (5,18)\ (7,13)\ (8,16)\ (9,11)\ (10,19)\ (12,17) \quad : 2^{10}$$
$$y = (1,3,15)\ (4,7,14)\ (5,18,6)\ (8,17,13)\ (9,12,16)\ (10,19,11)\ \underline{0}\ \underline{2} \quad : 3^{6}.1^{2}$$
$$z = xy = (0,1,2,\ldots\quad ,15)\ (16,17)\ \underline{18}\ \underline{19} \quad : 16.2.1^{2}$$
$$\sigma = yz^{4} = (0,4,11,14,8,17,1,7,2,6,9)\ (5,18,10,19,15)\ (12,16,13)\ \underline{3} \quad : 11.5.3.1$$

y^{z} and σ show that H is 2-transitive. Hence, $H = A_{20}$, since $\sigma \in H$.

Lemma 2.4

A_{21} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$$x = (1,3)\ (2,18)\ (4,15)\ (5,7)\ (6,19)\ (8,14)\ (9,17)\ (10,12)\ (11,20)\ (13,16)\ \underline{0} \quad : 2^{10}.1$$
$$y = (0,1,4)\ (2,18,3)\ (5,8,15)\ (6,19,7)\ (9,16,14)\ (10,13,17)\ (11,20,12) \quad : 3^{7}$$
$$z = xy = (0,1,2,\ldots\quad ,15)\ (16,17)\ \underline{18}\ \underline{19}\ \underline{20} \quad : 16.2.1^{3}$$
$$\sigma = yz^{4} = (0,5,12,15,9,16,2,18,7,10,1,8,3,6,19,11,20)\ (13,17,14)\ \underline{4} \quad : 17.3.1$$

x^{z^4} and σ show that H is 2-transitive. Hence, $H = A_{21}$ since $\sigma \in H$.

Lemma 2.5

A_{22} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$$x = (2,15)\ (3,17)\ (4,13)\ (5,19)\ (6,11)\ (7,21)\ (8,9)\ (10,20)\ (12,18)\ (14,16)\ \underline{0}\ \underline{1} \quad : 2^{10}.1^{2}$$
$$y = (0,1,2)\ (3,16,15)\ (4,14,17)\ (5,18,13)\ (6,12,19)\ (7,20,11)\ (8,10,21)\ \underline{9} \quad : 3^{7}.1$$
$$z = xy = (0,1,2,\ldots\quad ,15)\ (16,17)\ (18,19)\ (20,21) \quad : 16.2^{3}$$
$$\sigma = yz^{4} = (0,5,18,1,6)\ (2,4)\ (3,16)\ (7,20,15)\ (8,14,17)\ (9,13)\ (10,21,12,19)\ \underline{11} \quad : 5.4.3^{2}.2^{3}.1$$

Now σ and y^{z^2} show that H is 2-transitive. Hence, $H = A_{22}$, since $\sigma \in H$.

Lemma 2.6

$A_{23} \notin (2,3,16)$-group.

Proof

Application of (II) shows that $g < 0$. So, there is no transitive (2,3,16)-subgroup of A_{23}.

Lemma 2.7

A_{24} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (2,15)\ (4,14)\ (6,13)\ (8,12)\ (9,23)\ (10,18)\ (11,16)\ (9,22)\ \underline{0}\ \underline{1}\ \underline{3}\ \underline{5}\ \underline{7}\ \underline{17}\ \underline{20}\ \underline{21}$: $2^8.1^8$

$y = (0,1,2)\ (3,4,15)\ (5,6,14)\ (7,8,13)\ (9,16,12)\ (10,19,23)\ (11,17,18)\ (20,21,22)$: 3^8

$z = xy = (0,1,2,\ldots\ ,15)\ (16,17,\ldots\ ,23)$: 16.8

$\sigma = yz^4 = (0,5,10,23,14,9,20,17,22,16)\ (1,6,2,4,3,8)\ (7,12,13,11,21,18,15)\ \underline{19}$: $10.7.6.1$

x and σ^z show that H is 2-transitive. Hence, $H = A_{24}$, since $\sigma\ \varepsilon\ H$.

Lemma 2.8

A_{25} is a (2,3,16)-group.

Proof

Let $H = \langle x, y \rangle$, where:

$x = (1,23)\ (2,17)\ (3,14)\ (4,5)\ (6,13)\ (7,19)\ (8,21)\ (9,11)\ (10,24)\ (12,20)\ (15,16)\ (18,22)\ \underline{0}$: $2^{12}.1$

$y = (0,1,16)\ (2,18,23)\ (3,15,17)\ (4,6,14)\ (7,20,13)\ (8,22,19)\ (9,12,21)\ (10,24,11)\ \underline{5}$: $3^8.1$

$z = xy = (0,1,2,\ldots\ ,15)\ (16,17,\ldots\ ,23)\ \underline{24}$: $16.8.1$

$\sigma = yz^3 = (0,4,9,15,20)\ (1,19,11,13,10,24,\underline{14}\ 7,23,5,8,17,6)\ (2,21,12,16,3)\ \underline{18}\ \underline{22}$: $13.5^2.1^2$

H is primitive, since it has no blocks of length 5. Hence, $H = A_{25}$ because $\sigma\ \varepsilon\ H$.

References

AL-Salman, S.A.M. (1973) *On Some (2,3,n)-Groups*, PH.D. Thesis, Univ. of Birmingham, pp 106-110 (unpublished).

Coxeter, H.S.M. and **Moser, W.O.J.** (1980) *Generators and Relations for Discrete Groups*, 4^{th} edn, Berlin, pp 61-82.

Singerman, D. (1970) Subgroups of Fuchsian groups and finite permutation groups, *Bull. London Math. Soc.* **2**: 319-323.

Wielandt, H. (1968) *Finite Permutation Groups*, 2^{nd} edn, Academic Press Inc., pp 1-40.

(*Received 17/12/1983;*
in revised form 23/05/1984)