Arab Gulf J. scient. Res. **3** (2), pp. 625-630 (1985)

On Some (2,3,16)-Groups of Degree N, $16 \leq N \leq 30$

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ABSTRACT. This is a sequel to an earlier paper by the author in which it was proved that the symmetric group S_N of degree N is a (2,3,16)-group for $16 \leq N \leq 25$ whereas the alternating group A_N is a (2,3,16)-group for $18 \leq N \leq 25$, $N \neq 23$. In this paper, it is proved that both S_N and A_N are (2,3,16)-groups for $26 \leq N \leq 30$.

Proposition 1

The symmetric group S_N is a (2,3,16)-group for $16 \leq N \leq 30$.

Proof

The proof will be provided as a series of lemmas.

Lemma 1.1

S_N is a (2,3,16)-group for $16 \leq N \leq 25$ (Al-Salman 1985).

Lemma 1.2

S_{26} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (0,15)(1,2)(3,14)(4,25)(5,12)(6,23)(8,22)(10,21)(11,16)(13,24)(17,20)\ \underline{7}\ \underline{9}\ \overline{18}\ \overline{19}$ $: 2^{11}.1^4$

$y = (1,3,15)(4,24,14)(5,13,25)(6,16,12)(7,8,23)(9,10,22)(11,17,21)(18,19,20)\ \underline{0}\ \underline{2}$
$: 3^8.1^2$

$z = xy = (0,1,2,\ldots\ ,15)(16,17,\ldots\ ,23)(24,25)$ $: 16.8.2$

$\sigma = xz^4 = (0,3,2,5)(1,6,19,23,10,17,16,15,4,25,8,18,22,12,9,13,24)(7,11,20,21,14)$
$: 17.5.4$

Now x and σ^5 show that H is 2-transitive. It then follows that $H = S_{26}$, since $\sigma \in H$.

Lemma 1.3

S_{27} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (0,26)(1,15)(2,23)(4,22)(5,17)(6,13)(7,25)(8,11)(9,19)(10,20)(12,24)(14,16)$
$(18,21)\ \underline{3}$
$: 2^{13}.1$

$y = (0,26,1)(2,16,15)(3,4,23)(5,18,22)(6,14,17)(7,24,13)(8,12,25)(9,20,11)$
$(10,21,19)$
$: 3^9$

$z = xy = (0,1,2,\ldots\ ,15)(16,17,\ldots,23)(24,25)\ \underline{26}$ $: 16.8.2.1$

$\sigma = xz^3 = (0,26,3,6)(1,2,18,16)(4,17,8,14,19,12,25,10,23,5,20,13,9,22,7,24,15)$
$\underline{11}\ \underline{21}$
$: 17.4^2.1^2$

x^{z^8} and σ indicate the H is 2-transitive. Hence $H = S_{27}$, since $\sigma \in H$.

Lemma 1.4

S_{28} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (0,23)\ (1,24)\ (2,16)\ (3,15)\ (5,14)\ (6,7)\ (8,13)\ (9,18)\ (10,26)\ (11,21)\ (12,19)$
$(17,27)\ (22,25)\ \underline{4}\ \underline{20}$
$: 2^{13}.1^2$

$y = (0,16,3)\ (1,25,23)\ (2,17,24)\ (4,5,15)\ (6,8,14)\ (9,19,13)\ (10,27,18)\ (11,22,26)$
$(12,20,21)\ \underline{7}$
$: 3^9.1$

$z = xy = (0,1,2,\ldots\ ,15)\ (16,17,\ldots\ 23)\ (24,25,26,27)$ $: 16.8.4$

$\sigma = xz^3 = (0,18,12,22,24,4,7,9,21,14,8)\ (1,27,20,23,3,2,19,15,6,10,25,17,26,13,11,$
$16,5)$
$: 17.11$

Now y and σ^{11} show that H is 2-transitive. It then follows that $H = S_{28}$, since $\sigma \in H$.

Lemma 1.5

S_{29} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (0,28)\ (1,15)\ (2,27)\ (4,26)\ (5,11)\ (6,23)\ (8,22)\ (9,19)\ (10,16)\ (12,25)\ (14,24)$
$\quad (17,18)\ (20,21)\ \underline{3}\ \underline{7}\ \underline{13}$ $\quad : 2^{13}.1^3$
$y = (0,28,1)\ (2,24,15)\ (3,4,27)\ (5,12,26)\ (6,16,11)\ (7,8,23)\ (9,20,22)\ (10,17,19)$
$\quad (13,14,25)\ \underline{18}\ \underline{21}$ $\quad : 3^9.1^2$
$z = xy = (0,1,2,\ldots\ ,15)\ (16,17,\ldots,23)\ (24,25,26,27)\ 28$ $\quad : 16.8.4.1$
$\sigma = xz^3 = (0,28,3,6,18,20,16,13)\ 1,2,26,7,10,19,12,24)\ (4,25,15)\ (5,14,27)$
$\quad (8,17,21,23,9,22,11)$ $\quad : 8^2.7.3^2$

$H = S_{29}$, since H is primitive and $\sigma \in H$.

Lemma 1.6

S_{30} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (0,28)\ (1,15)\ (2,27)\ (3,25)\ (4,13)\ (5,23)\ (6,21)\ (7,\ 19)\ (8,17)\ (9,11)\ ((10,29)$
$\quad (12,16)\ (14,24)\ \underline{18}\ \underline{20}\ \underline{22}\ \underline{26}$ $\quad : 2^{13}.1^4$
$y = (0,28,1)\ (2,24,15)\ (3,26,27)\ (4,14,25)\ (5,16,13)\ (6,22,23)\ (7,20,21)\ (8,18,19)$
$\quad (9,12,17)\ (10,29,11)$ $\quad : 3^{10}$
$z = xy = (0,1,2,\ldots\ ,15)\ (16,17,\ldots,23)\ (24,25,26,27)\ \underline{28}\ \underline{29}$ $\quad : 16.8.4.1^2$
$\sigma = xz^4 = (0,28,4,1,3,25,7,23,9,15,5,19,11,13,8,21,10,29,14,24,2,27,6,17,12,20,16)$
$\quad (18,22)\ \underline{26}$ $\quad : 27.2.1$

Now x^z and $(xz^7)^2$ show that H is 2-transitive. It then follows that $H = S_{30}$, since $\sigma \in H$.

Proposition 2

The alternating group A_N is a (2,3,16)-group for $N \neq 23$ and $18 \leqslant N \leqslant 30$.

Proof

The proof will be provided as a series of lemmas.

Lemma 2.1

A_N is a (2,3,16)-group for $N \neq 23$ and $18 \leqslant N \leqslant 25$ (Al-Salman 1985).

Lemma 2.2

A_{26} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (2,15)\ (3,11)\ (4,23)\ (5,20)\ (6,17)\ (7,9)\ (8,25)\ (10,16)\ (12,14)\ (13,24)\ (18,19)\ (21,22)\ \underline{0}\ \underline{1}$ $\quad : 2^{12}.1^2$

$y = (0,1,2)\ (3,12,15)\ (4,16,11)\ (5,21,23)\ (6,18,20)\ (7,10,17)\ (8,25,9)\ (13,24,14)\ \underline{19}\ \underline{22}$ $\quad : 3^8.1^2$

$z = xy = (0,1,2,\ldots,15)\ (16,17,\ldots,23)\ \underline{24}\ \underline{25}$ $\quad : 16.8.1^2$

$\sigma = xz^8 = (0,8,25)\ (1,9,15,10,16,2,7)\ (4,23,12,6,17,14)\ (5,20,13,24)\ (18,19)\ (21,22)\ \underline{3}\ \underline{11}$ $\quad 7.6.4.3.2^2.1^2$

Now x^{z^2} and σ show that H is 2-transitive. It then follows that $H = A_{26}$, since $\sigma \in H$.

Lemma 2.3

A_{27} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (0,26)\ (1,15)\ (2,23)\ (4,22)\ (5,18)\ (6,9)\ (10,17)\ (11,13)\ (12,24)\ (14,16)\ (19,21)\ (20,25)\ \underline{3}\ \underline{7}\ \underline{8}$ $\quad : 2^{12}.1^3$

$y = (0,26,1)\ (2,16,15)\ (3,4,23)\ (5,19,22)\ (6,10,18)\ (7,8,9)\ (11,14,17)\ (12,24,13)\ (20,25,21)$ $\quad : 3^9$

$z = xy = (0,1,2,\ldots\ ,15)\ (16,17,\ldots,23)\ \underline{24}\ \underline{25}\ \underline{26}$ $\quad : 16.8.1^3$

$\sigma = xz^6 = (0,26,6,15,7,13,1,5,16,4,20,25,18,11,3,9,12,24,2,21,17)\ (8,14,22,10,23)\ \underline{19}$ $\quad : 21.5.1$

x and σ^5 indicate that H is 2-transitive. Hence, $H = A_{27}$, since $\sigma \in H$.

Lemma 2.4

A_{28} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$x = (0,24)\ (1,15)\ (2,23)\ (3,5)\ (4,25)\ (6,22)\ (7,8)\ (9,21)\ (10,17)\ (11,13)\ (12,26)\ (14,16)\ (18,20)\ (19,27)$ $\quad : 2^{14}$

$y = (0,24,1)\ (2,16,15)\ (3,6,23)\ (4,25,5)\ (7,9,22)\ (10,18,21)\ (11,14,17)\ (12,26,13)\ (19,27,20)\ \underline{8}$ $\quad : 3^9.1$

$z = xy = (0,1,2,\ldots\ 15)\ (16,17,\ldots,23)\ \underline{24}\ \underline{25}\ \underline{26}\ \underline{27}$ $\quad : 16.8.1^4$

$\sigma = xz^6 = (0,24,6,20,16,4,25,10,23,8,13,1,5,9,19,27,17)\ (2,21,15,7,14,22,12,26)\ (3,11)$ $\quad : 17.8.2$

y^{z^3} and σ^2 show that H is 2-transitive. Hence $H = A_{28}$, since $\sigma \in H$.

Lemma 2.5

A_{29} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$$x = (0,28)\ (1,15)\ (2,25)\ (3,13)\ (4,27)\ (5,11)\ (6,23)\ (8,22)\ (9,19)\ (10,16)\ (12,26)\ (14,24)\ (17,18)\ (20,21)\ \underline{7} \qquad : 2^{14}.1$$

$$y = (0,28,1)\ (2,24,15)\ (3,14,25)\ (4,26,13)\ (5,12,27)\ (6,16,11)\ (7,8,23)\ (9,20,22)\ (10,17,19)\ \underline{18}\ \underline{21} \qquad : 3^9.1^2$$

$$z = xy = (0,1,2,\ldots,15)\ (16,17,\ldots,23)\ (24,25)\ (26,27)\ \underline{28} \qquad : 16.8.2^2.1$$

$$\sigma = xz^3 = (0,28,3)\ (1,2,24)\ (4,26,15)\ (5,14,25)\ (6,18,20,16,13)\ (7,10,19,12,27)\ (8,17,21,23,9,22,11) \qquad : 7.5^2.3^4$$

$H = A_{29}$, since H is primitive and $\sigma \in H$.

Lemma 2.6

A_{30} is a (2,3,16)-group.

Proof

Let $H = \langle x,y \rangle$, where:

$$x = (10,29)\ (2,28)\ (3,15)\ (4,23)\ (6,22)\ (8,21)\ (10,20)\ (11,18)\ (12,26)\ (13,24)\ (14,16)\ (17,27)\ \underline{1}\ \underline{5}\ \underline{7}\ \underline{9}\ \underline{19}\ \underline{25} \qquad : 2^{12}.1^6$$

$$y = (0,28,3)\ (1,2,29)\ (4,16,15)\ (5,6,23)\ (7,8,22)\ (9,10,21)\ (11,19,20)\ (12,27,18)\ (13,25,26)\ (14,17,24) \qquad : 3^{10}$$

$$z = xy = (0,1,2,\ldots,15)\ (16,17,\ldots,23)\ (24,25,26,27)\ (28,29) \qquad : 16.8.4.2$$

$$\sigma = xz^4 = (0,29,4,19,23,8,17,27,21,12,26)\ (1,5,9,13,24)\ (2,28,6,18,15,7,11,22,10,16)\ (14,20)\ \underline{3}\ \underline{25} \qquad : 11.10.5.2.1^2$$

Now x and $(xz^6)^2$ show that H is 2-transitive. It then follows that $H = A_{30}$, since $\sigma \in H$.

References

Al-Salman, S.A. (1985) On some (2,3,16)-groups of degree N, $16 \leq N \leq 25$, *Arab Gulf J. Scient. Res.* **3**: 249-258.

(Received 16/10/1984;
in revised form 05/01/1985)

بعض الزمر ذوات المولدين (2, 3, 16) من الدرجة N, $16 \leqslant n \leqslant 30$

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هذا البحث هو استمرار لبحث قبله للمؤلف نفسه والذى برهن فيه أن كل زمرة من زمر التناظر $S_N (16 \leqslant N \leqslant 25)$ يمكن توليدها بعنصرين x, y بحيث $x^2 = y^3 = (xy)^{16} = 1$ وكذلك كل زمـــرة متنـــاوبـة $A_N (18 \leqslant N \leqslant 25, N \neq 23)$ يمكن توليدهـا بعنصرين أيضاً x, y بحيث:

$$x^2 = y^3 = (xy)^{16} = 1$$

وفى هذا البحث برهنّـا أن كلاً من زمـرة التنـاظر S_N وزمرة التناوب A_N $(26 \leqslant N \leqslant 30)$ يمكن توليدها بعنصرين x, y بحيث:

$$x^2 = y^3 = (xy)^{16} = 1$$