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Fixed Point Theory in Bitopological Spaces

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ABSTRACT. The relation between any bitopological space and its least upper bound topology is studied. A new type of pairwise continuous functions between bitopological spaces is presented as well. Then, we use the concept of pairwise continuous functions to introduce the concept of pairwise fixed point property, pairwise retraction functions and pairwise contraction functions between bitopological spaces. We also obtain some generalizations of some well known results concerning fixed point theory for a single topological space.

In 1963, Kelly introduced the notion of a bitopological space which is the triple (X, τ_1, τ_2), where X is a set and τ_1 and τ_2 are two topologies on X. Later on, several authors had studied this notion and other related concepts. In particular, Pervin (1967) defined connectedness properties for bitopological spaces. Birsan (1969), Reilly (1970), and Swart (1971) discussed various aspects of connectedness properties. Kelly (1963) also defined pairwise Hausdorff, pairwise regular and pairwise normal spaces, then he obtained some generalizations of several standard results such as Urysohn's Lemma and Tietze's extension theorem.

We shall use p-, s- to denote pairwise, semi-, respectively, *e.g.* p-compact, s-compact stand for pairwise compact and semi-compact respectively. If A is a subset of the topological space (X, τ), then the relative topology on the set A inherited by τ will be denoted by τ_A. The product topology of τ_1 and τ_2 will be denoted by $\tau_1 x\ \tau_2$. Let τ_L, τ_r, τ_d, τ_u, τ_f denote the left ray, right ray, discrete, usual and the cofinite topology on R, respectively.

1.1 *Definition.* When we say that a bitopological space (X, τ_1, τ_2) has a particular topological property, without referring specially to τ_1 or τ_2, we mean that both τ_1 and τ_2 have the property, for instance, (X, τ_1, τ_2) is said to be Hausdorff if both (X, τ_1) and (X, τ_2) are Hausdorff.

1.2 *Definition.* A function $f: X \rightarrow X$ has a fixed point if there exists a point $t \varepsilon X$ such that $f(t) = t$. The point t is called a fixed point of f. A topological space (X, τ) has the fixed point property (abbreviated by f.p.p.) if every continuous function from X into itself has a fixed point.

1.3 *Definition.* Let f be a function from a metric space (X, d) into itself. We say that f is a contraction function if there exists a real number $o \leqslant \lambda < 1$ such that:

$$d\ (f(x),\ f(y)) \leqslant \lambda\ d(x,\ y),\ \text{for all } x,y\ \varepsilon\ X.$$

In 1922 Banach proved that "any contraction mapping of a complete non-empty metric space X into itself has a unique fixed point in X". In this paper we study this theorem in bimetric spaces and give some results.

1.4 *Definition.* A continuous function r from a space X onto a subspace A of X is called a retraction function if the restriction function $r|A$ is the identity function on A. When such a retraction exists, A is called a retract of X.

The following are well-known results.

1.5 *Proposition.* Every retract of a space with the f.p.p. has the f.p.p.

1.6 *Proposition.* If a space X has the f.p.p., then X must be connected.

1.7 *Proposition.* If a space X has the f.p.p, then X must be a T_o-space.

In this paper we give a generalization of these propositions in bitopological spaces. To proceed we give the following definitions.

1.8 *Definition.* Let (X, τ_1, τ_2) be any bitopological space. If there exist non empty sets $U,V\ \varepsilon \tau_1 U\ \tau_2$ such that $U \cap V = \phi$ and $U \cup V = X$ then (X, τ_1, τ_2) is called s-disconnected. A bitopological space (X, τ_1, τ_2) is called s-connected if it is not s-disconnected.

1.9 *Definition.* A bitopological space (X, τ_1, τ_2) is called a p-T_0 -space iff for every pair of distinct points, there exists a τ_1- or aτ_2-open set which contains one point but not the other.

2. *The Least Upper Bound Topology*

In this section we shall investigate the relation between the least upper bound topology and bitopological spaces. To proceed we need the following definition.

2.1 *Definition.* Let τ_1 and τ_2 be two topologies on X. Then $\tau_1 \cup \tau_2$ forms a subbase for some topology on X. This topology is called the least upper bound topology on X, and is denoted by $< \tau_1, \tau_2 >$.

The following result clarifies the relation between (X, τ_1), (X, τ_2) and $(X, < \tau_1, \tau_2 >)$.

2.2 *Theorem.* Let (X, τ_1, τ_2) be a bitopological space and let $\triangle = \{(x,x) : x \varepsilon X\}$ be the diagonal subspace of $(X \times X, \tau_1 \times \tau_2)$. Then $(X, < \tau_1, \tau_2 >)$ is homeomorphic to $\triangle$.

Proof. Define f: $(X, < \tau_1, \tau_2 >) \to \triangle$ by $F(x) = (x,x)$. It is clear that f is a bijection. Since $f(U \cap V) = (U \times V) \cap \triangle$ is true for all $U \varepsilon \tau_1$, $V \varepsilon \tau_2$, therefore it is easy to check that f is continuous and open. Hence f is a homeomorphism.

Now, it is easy to observe the following corollary.

2.3 *Corollary.* If (X, τ_1, τ_2) satisfies the property P, then $(X, < \tau_1, \tau_2 >)$ satisfies P, where P is one of the following: a. regular, b. completely regular, c. second countable, d. first countable, e. metrizable.

3. *Fixed Point Theory in Bitopological Spaces*

Let us start this section with the following definitions.

3.1 *Definition.* Let f: $(X, \tau_1, \tau_2) \to (Y, \tau'_1, \tau'_2)$ be a function.

(i) If f is continuous as a function from (X, τ_1) into (Y, τ'_1) and f is continuous as a function from (X, τ_2) into (Y, τ'_2). Then f is called a continuous function.

(ii) If for each $U \varepsilon \tau'_1 \cup \tau_2'$, the inverse image of $U, f^{-1}(U) \varepsilon \tau_1 \cup \tau_2$, then f is called a p-continuous function.

3.2 *Definition.* Let (X, τ_1, τ_2) be a bitopological space.

(i) If every continuous function from (X, τ_1, τ_2) into itself has a fixed point, then we say that X has the f.p.p.

(ii) If every p-continuous function from (X, τ_1, τ_2) into itself has a fixed point, then we say that X has the p-f.p.p.

3.3 *Theorem.* If (X, τ_1, τ_2) has the p-f.p.p., then X has the f.p.p.

The proof of this theorem is easy because every continuous function is p-continuous.

3.4 *Lemma.* If f is a p-continuous function from (X, τ_1, τ_2) into (Y, τ'_1, τ'_2), then f is continuous as a function from $(X, < \tau_1, \tau_2 >)$ into $(Y, < \tau'_1, \tau'_2 >)$.

Proof: Let U be any subbasic open set in $(Y, <\tau'_1, \tau'_2>)$, then $U \varepsilon \tau'_1 \cup \tau'_2$ and so $f^{-1}(U) \varepsilon \tau_1 \cup \tau_2$. But $\tau_1 \cup \tau_2 \subseteq <\tau_1, \tau_2>$, therefore f is a continuous function from $(X, <\tau_1, \tau_2>)$ into $(Y, <\tau'_1, \tau'_2>)$.

Now using this lemma we can easily prove the following theorem.

3.5 *Theorem.* If (X, τ_1, τ_2) is a bitopological space such that $(X, <\tau_1, \tau_2>)$ has the f.p.p. Then (X, τ_1, τ_2) has the p-f.p.p.

The following example shows that the converse of the above theorem is in general false.

3.6 *Example.* There exists a bitopological space (X, τ_1, τ_2) for which (X, τ_1, τ_2) has the p-f.p.p. and $(X, <\tau_1, \tau_2>)$ does not have the f.p.p.

Proof. Let $X = \{1,2,3,4\}$, $\tau_1 = \{\phi, X, \{1,2\}\}$, and $\tau_2 = \{\phi, X, \{3\}, \{1,3\}, \{2,3,4\}\}$.

Then $<\tau_1, \tau_2> = \{\phi, X, \{1,2\}, \{3\}, \{1,3\}, \{2,3,4\}, \{1\}, \{2\}\}$. Since $\{1\}$ is closed and open in the topological space $(X, <\tau_1, \tau_2>)$, therefore $(X, <\tau_1, \tau_2>)$ does not have the f.p.p. After doing some calculations, one can check that (X, τ_1, τ_2) has indeed the p-f.p.p.

3.7 *Theorem. If (X, τ_1, τ_2) has the p-f.p.p., then X is s-connected.*

Proof. Suppose that (X, τ_1, τ_2) is s-disconnected. Then, there exist non-empty sets $U, V \varepsilon \tau_1 \cup \tau_2$ such that $U \cap V = \phi$ and $U \cup V = X$. Let $p \varepsilon U$ and $q \varepsilon V$, define $f: X \to X$ by $f(V) = p$ and $f(U) = q$. Then, f is a p-continuous function but does not have any fixed point; a contradiction.

3.8 *Theorem.* If (X, τ_1, τ_2) has the p-f.p.p., then X is a p-T_0 -space.

Proof. Suppose that X is not a p-T_0 -space, then there exist two distinct elements in X, say p and q, such that there is no $U \varepsilon \tau_1 \cup \tau_2$ which contains p or q but not both; *i.e.* every $U \varepsilon \tau_1 \cup \tau_2$ which contains p must contain q and *vice versa.* Define $f: X \to X$ by $f(x) = p$ for all $x \neq p$ and $f(p) = q$. Then, the inverse image of any member of $\tau_1 \cup \tau_2$ is ϕ or X, because every member of $\tau_1 \cup \tau_2$ must contain both p and q or contains neither p nor q. Therefore, f is a p-continuous function but does not have any fixed point; a contradiction.

It is important to observe that Theorem 3.8 can not be improved to pairwise T_1. For if $X = \{1,2\}$, $\tau_1 = \tau_2 = \{\phi, X, \{1\}\}$, then (X, τ_1, τ_2) has the p-f.p.p. but X is not a p-T_1-space.

Now we shall introduce the definition of pairwise retraction function and pairwise retract in any bitopological space, then we obtain some related results.

3.9 *Definition.* Let (X, τ_1, τ_2) be a bitopological space and let A be any subset of X.

(i) If there exists a function $r:(X, \tau_1, \tau_2) \to (A, \tau_{1A}, \tau_{2A})$ which is p-continuous and such that $r|A = i_A$. Then A is called a p-retract of X and r is called p-retraction.

(ii) If A is a retract of (X,τ_1) and a retract of (X,τ_2). Then A is called a retract of X.

The following two examples show that there is no relation between p-retraction and retraction.

3.10 *Example.* In the bitopological space (R, τ_d, τ_u), the set $A = (-1, 1)$ is a p-retract but it is not a retract of (R, τ_u). Thus, it is not a retract of (R, τ_d, τ_u).

3.11 *Example.* Let $X = (0, \infty)$, $\tau_1 = \tau_f$, $\tau_2 = \tau_r$ and $A = (0, 1]$.
Define $r: X \to A$ by $r(x) = x$ for all $x \varepsilon A$ and $r(x) = 1/x$ for all $x \varepsilon [1, \infty)$. Since every element of A is an image of at most two elements, therefore r is continuous as a function from (X, τ_1) onto (A, τ_{1A}). Therefore, A is a τ_1-retract of X.

Let $r': X \to A$ be defined by $r'(x) = x$ for all $x \varepsilon A$ and $r'(x) = 1$ otherwise. Then, r' is increasing. Consequently, r' is continuous as a function from (X, τ_2) onto (A, τ_{2A}). Hence A is a τ_2 -retract of X. Thus, A is a retract of (X, τ_1, τ_2).

To show that A is not a p-retract of (X, τ_1, τ_2), suppose on the contrary that A is a p-retract of (X, τ_1, τ_2). Then, there exists a p-continuous function $h: X \to A$ such that $h|A = i_A$. If $x \varepsilon A$, then $(x, 1] \varepsilon \tau_{2A}$, thus $h^{-1}((x,1]) \varepsilon \tau_1 \cup \tau_2$. Therefore, $h^{-1}((x,1]) \varepsilon \tau_2$ because $(0,x]$ is an infinite set and $h^{-1}((0,x])$ is infinite. Hence, h is continuous as a function from (X, τ_2) onto (A, τ_{2A}). Since τ_{2A} is the right ray topology on A and h is continuous, therefore h is an increasing function. Hence, $h = r'$ is the only possible way. But, $(0, 1) \varepsilon \tau_{1A}$ and $h^{-1}(0,1) = (0,1) \not\varepsilon \tau_1 \cup \tau_2$. Therefore, h is not p-continuous; a contradiction.

3.12 *Theorem.* If A is a p-retract of (X, τ_1, τ_2), then A is a retract of $(X, < \tau_1, \tau_2 >)$.

The proof of this Theorem is easy by using Lemma 3.4.

3.13 *Corollary.* Let (X, τ_1, τ_2) be a p-Hausdorff space, and let A be a p-retract of X. Then, A is a closed subset of $(X, < \tau_1, \tau_2 >)$.

The following example shows that the converse of the above theorem is in general false.

3.14 *Example.* There exists a bitopological space (X, τ_1, τ_2) and a set $A \subset X$ for which A is a retract of $(X, <\tau_1, \tau_2>)$ but A is not a p-retract of (X, τ_1, τ_2).

Proof. Let $X = \{1,2,3,4\}$, $\tau_1 = \{\phi, X, \{1,2\}, \{3,4\}\}$, $\tau_2 = \{\phi, X, \{1,3\}, \{2,4\}\}$, and $A = \{1,2,3\}$. Then, $\tau_{1,A} = \{\phi, A, \{1,2\}, \{3\}\}$, $\tau_{2,A} = \{\phi, A, \{1,3\}, \{2\}\}$, and $<\tau_1, \tau_2> = \tau_{d,X}$. It is clear that A is a retract of $(X, <\tau_1, \tau_2>)$ because $<\tau_1, \tau_2>$ is the discrete topology on X. After doing simple calculations, one can check that A is indeed not a p-retract of (X, τ_1, τ_2).

3.15 *Theorem.* If the bitopological space (X, τ_1, τ_2) has the p-f.p.p., then every p-retract of X has the p-f.p.p.

Proof. Let A be a p-retract of X and let $r: X \to A$ be a p-retraction function. Let $f: A \to A$ be any p-continuous function, then $(f \circ r): X \to A$ is a p-continuous function from X into itself. Thus, $(f \circ r)$ has a fixed point in A because $(f \circ r)(X) = A$, *i.e.* there exists $p \varepsilon A$ such that $p = (f \circ r)(p) = f(p)$. Hence A has the p-f.p.p.

4. *Pairwise Complete Bimetric Spaces and Pairwise Contraction Functions*

In this section, we shall define the concept of pairwise complete bimetric spaces and the concept of pairwise contraction functions, then we obtain some related results and an analogue of Banach's theorem in bitopological spaces.

4.1 *Definition.* Let (X, d_1, d_2) be a bimetric space. Then, X is called (i,j)-complete if every d_i-Cauchy sequence has a d_j-limit point $(i \neq j,\ i,j = 1,2)$.

If (X, d_1, d_2) is (1,2)-complete and (2,1)-complete then (X,d_1,d_2) is called p-complete.

The following example shows that (X,d_1,d_2) can be (1,2)-complete but neither d_1-nor d_2-complete.

4.2 *Example.* Let $X = [-1,1]$, and let d_1 be defined as follows:

$d_1(x,x) = 0$ for all $x \varepsilon X$, $d_1(x,y) = 1$ for all $y \varepsilon X$, $y \neq x, x \varepsilon [0,1]$ and $d_1(x,y) = |x-y|$ for all $x,y \varepsilon [-1,0)$. Let d_2 be defined as follows:

$d_2(x,x) = 0$ for all $x \varepsilon [-1,1]$, $d_2(x,y) = |x-y|$ for all $x,y \varepsilon [-1,1)$ and $d_2(1,x) = 1$ for all $x \varepsilon [-1,1)$.

Then, the sequence $x_n = (n-1)/n$ is a d_2-Cauchy sequence but does not have any d_2-limit point, therefore (X,d_2) is not complete. The sequence $x_n = -1/n$ is also a d_1-Cauchy sequence but does not have any d_1-limit point. Therefore (X,d_1) is not complete.

However if $(x_n)_1^\infty$ is any d_1-Cauchy sequence then we have two cases to consider:

Case I: The tail of $(x_n)_1^\infty$ is constant, then $(x_n)_1^\infty$ is convergent in (X,d_2).

Case II: The tail of $(x_n)_1^\infty$ is not constant, then a tail of (x_n) is contained in $[-1,0)$; but $[-1,0)$ has the usual metric as d_1 and $([-1,0], d_2)$ is the usual metric space so it is complete, therefore $(x_n)_1^\infty$ has a d_2-limit point. Hence (X,d_1,d_2) is (1,2)-complete.

4.3 *Theorem.* Every p-complete bimetric space is complete.

Proof: Let $(x_n)_1^\infty$ be a d_i-Cauchy sequence. Then (x_n) is a d_j-convergent, $i \neq j$. But every convergent sequence in a metric space is a Cauchy sequence, therefore (x_n) is a d_j-Cauchy sequence, so it has a d_i-limit point. Hence (X,d_i) is complete; $i = 1,2$.

4.4 *Example.* Let (R,d_u) and (R,d_d) denote the usual metric space and the discrete metric space on R, respectively. Then, (R,d_u) and (R, d_d) are complete metric spaces but the sequence $(1/n)_1^\infty$ is a d_u-Cauchy but does not have any d_d-limit point. Therefore (R, d_u,d_d) is not p-complete.

4.5 *Theorem.* Let (X, d_1,d_2) be a (1,2)-complete bimetric space, and let $f:X\to X$ be a d_1-contraction and d_2-continuous. Then, f has a unique fixed point.

Proof: Let $x_0 \varepsilon X$. If $f(x_0) = x_0$, then we are done. So, let $f(x_0) \neq x_0$ and then define $x_1 = f(x_0)$ and $x_{n+1} = f(x_n)$. It is easily seen that $(x_n)_1^\infty$ is a d_1-Cauchy sequence. Therefore, it has a d_2-limit point p. Since $(x_n)_1^\infty = (f(x_n)_0^\infty$, therefore the sequence $(f(x_n)_1^\infty d_2$-converges to p. But f is d_2-continuous, therefore $\lim f(x_n) = f(\lim x_n)$. Thus, $p = f(p)$.

For uniqueness, suppose that there exist two distinct elements x,y such that $f(x) = x$ and $f(y) = y$, then $d_1(x,y) = d_1(f(x), f(y)) \leqslant \lambda\, d_1(x,y)$, therefore $d_1(x,y) = 0$. Hence $x = y$; a contradiction.

4.6 *Definition.* The function $f: (X,d_1,d_2)\to(X,d_1,d_2)$ is called p-contraction if there exists $\lambda\varepsilon[0,1)$ such that

$d_1(f(x), f(y)) \leqslant \lambda\, d_2(x,y)$, and

$d_2(f(x), f(y)) \leqslant \lambda\, d_1(x,y)$ for all $x,y \varepsilon X$.

4.7 *Theorem.* Let f be a p-contraction function from the bimetric space (X,d_1,d_2) into itself. Then f is p-continuous.

Proof: Let $x \varepsilon X$ and $\varepsilon > 0$ be any real number, and let $B_i(f(x),\varepsilon)$ denote the d_i-open ball with center f(x) and radius ε. We claim that

$$f(B_i(x,\varepsilon)) \subseteq B_j(f(x),\varepsilon);\ i \neq j, i,j = 1,2.$$

To prove our claim, let $y\ \varepsilon B_i(x,\varepsilon)$. Then $d_i(x,y) < \varepsilon$. Therefore, $d_j(f(x), f(y)) \leqslant \lambda d_i(x,y) < \lambda\varepsilon < \varepsilon$. Hence, f is a p-continuous function.

Now, we are ready to give the following lemma which will be needed in the proof of the next theorem.

4.8 *Lemma.* Let (X,d_1,d_2) be a p-complete bimetric space. Then (X,d_1+d_2) is a complete metric space.

Proof: It is clear that (X, d_1+d_2) is a metric space. To prove that (X, d_1+d_2) is complete. Let $(x_n)_1^\infty$ be any (d_1+d_2)-Cauchy sequence. Then (x_n) is a d_1- and d_2-Cauchy sequence because $d_1(x,y) \leqslant (d_1+d_2)(x,y)$ for all $x,y\ \varepsilon X$. Therefore $(x_n)_1^\infty$ has a d_1-limit point, say p, also $(x_n)_1^\infty$ has a d_2-limit point, say q. We claim that $p = q$. To prove our claim, consider the sequence (t_n); where $t_{2n-1} = x_n$ and $t_{2n} = p$; $(t_n)_1^\infty$ is a d_1-Cauchy sequence, so it has a d_2-limit point say z. Now (x_n) is a subsequence of (t_n), so $x_n \xrightarrow{d_2} z$. Thus, $z = q$. Also (p) is a subsequence of (t_n), so $p \xrightarrow{d_2} z$. Thus, $p = q$. It follows that $x_n \xrightarrow{d_1+d_2} P = q$.

The following example shows that $(X,d_1 + d_2)$ need not be complete even if (X,d_1) and (X,d_2) are complete.

4.9 *Example.* Let $X = [0,1] \cup \{2\}$, and let d_1 be defined as follows:

$d_1(x,x) = 0$ for all $x\varepsilon X$, $d_1(x,y) = |x-y|$ for all $x,y\varepsilon\ [0,1]$ and $d_1(x,2) = 1$ for all $x\varepsilon[0,1]$; and let d_2 be defined as follows:

$d_2(x,x) = 0$ for all $x\varepsilon X$, $d_2(x,y) = |x-y|$ for all $x,y\varepsilon\ [0,1)$, $d_2(1,x) = 1$ for all $x\varepsilon\{1\}$; and $d_2\ (2,x) = 1-x$ for all $x\varepsilon[0,1)$. Then, (X,d_1) and (X,d_2) are complete metric spaces. But, the sequence $(n/(n+1))_1^\infty$ is d_1-and d_2-Cauchy, so it is (d_1+d_2)-Cauchy, but does not have any (d_1+d_2)- limit point.

4.10 *Theorem.* Let (X,d_1,d_2) be a p-complete bimetric space. Then, every p-contraction function from X into itself has a unique fixed point.

Proof: Let $d = d_1+d_2$ (*i.e.* $d(a,b) = d_1(a,b) + d_2(a,b)$), then, d is a metric on X, and $d(f(x), f(y)) = d_1(f(x), f(y)) + d_2(f(x), f(y)) \leqslant \lambda d_1(x,y) + \lambda d_2(x,y) = \lambda (d_1(x,y) + d_2(x,y)) = \lambda d(x,y)$. This shows that f is a d-contraction function. Since (X,d_1,d_2) is p-complete, therefore; by Lemma 4.8; (X,d_1+d_2) is complete. Hence;

by Banach's Theorem f has a unique fixed point.

4.11 *Theorem.* Let $\lambda_1, \lambda_2 > 0$ be such that $(\lambda_1 \cdot \lambda_2) < 1$, and let f be a function from (X, d_1, d_2) into itself such that

$$d_1(f(x), f(y)) \leqslant \lambda_1 \, d_2(x,y) \text{ for all } x,y \varepsilon X \text{ and}$$
$$d_2(f(x), f(y)) \leqslant \lambda_2 \, d_1(x,y) \text{ for all } x,y \varepsilon X.$$

Then, f has a unique fixed point provided that (X, d_1) or (X, d_2) is a complete metric space.

Proof: Without loss of generality we may assume that (X, d_1) is a complete metric space. Let f^2 denote the composition function (fof), and let $x,y \varepsilon X$ be any two elements, then

$$d_1(f^2(x), f^2(y)) \leqslant \lambda_1 d_2(f(x), f(y)) \leqslant \lambda_1 \lambda_2 \, d_1(x,y).$$

This shows that f^2 is a d_1-contraction. Therefore there exists a unique element $x \varepsilon X$ such that $f^2(x) = x$. To show that $f(x) = x$, we have $(d_1(x,f(x)))\,(d_2(x,f(x))) = (d_1(f^2(x), f(x)))\,(d_2(f^2(x), f(x))) \leqslant (\lambda_1(d_2(f(x), x)))\,(\lambda_2(d_1(f(x), x))) = \lambda_1\lambda_2(d_1(f(x), (x)))\,(d_2(f(x), (x)))$. This shows that $(d_1(f(x),x)).\,(d_2(f(x),x)) = 0$ which implies that $d_1(f(x),x) = 0$ or $d_2(f(x), x) = 0$. In either case we have $x = f(x)$. Hence, f has a fixed point. To prove the uniqueness of such x, suppose that $x' \varepsilon X$ is a fixed point of f. Then $f^2(x') = f(f(x)')) = f(x') = x'$. Hence x' is a fixed point of f^2. Consequently $x' = x$.

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نظرية النقطة الثابتة في الفضاءات التبولوجية المزدوجة

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درسنا في هذا البحث العلاقة بين الفضاءات التبولوجية المزدوجة والفضاء «أصغر الفضاءات التبولوجية العلوية». كذلك عرفنا نوع جديد من الاقترانات ازدواجية الإتصال بين الفضاءات التبولوجية المزدوجة، ثم استخدمنا هذه الاقترانات لتعريف خاصية النقطة الثابتة المزدوجة، وخاصية التقليص المزدوجة، وغيرها من الخصائص المرتبطة بالفضاءات التبولوجية المزدوجة. وأخيراً استنبطنا بعض النظريات التي تعد تعميماً لبعض النظريات في نظرية النقطة الثابتة للفضاءات التبولوجية المنفردة.