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Mathematical Study of the Effect of Rock Deformation on its Porosity and Permeability

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ABSTRACT. The methods of mathematical modelling for the filtration properties of the rocks are discussed. The relation between the porosity of a rock and its clearance in underformed state, and under external loads is derived. The coefficient of volumetric filtration are expressed mathematically in two different cases; assuming the shape of the pores either cylindrical or slot shaped channels. The paper contains (37) mathematical formulae and three figures which illustrate various relationships between the studied parameters.

The porosity of a rock depends on the shape and size of the grains, degree of sorting, cementation and packing. It differs owing to the grain to grain relationship. It determines the possibility of leaching and melting of ores and migration of gases within the rock mass.

The movement of water within rocks causes solution, mechanical washing out, cementation and other processes. The ability of rocks to transmit fluids is characterized by the coefficient of permeability K_{pr} found from Darcy's equation, which was reviewed by Rzhevsky (1971), Roberts (1977), and Holtz and Kovacs (1981) and is expressed as follows:

$$V_p = K_{per}\,(\triangle F/\triangle L).(1/\eta) \qquad (1)$$

and

$$K_{per} = (Q/S).(\triangle L/\triangle F).\eta \qquad (2)$$

In mining, the coefficient of percolation (permeability) is mainly used as (K_p) where:

$$K_p = K_{per} (\gamma_w/\eta) \text{ m/sec.}$$

$$(\gamma_w/\eta) \text{ for water } = 1$$

Rocks are divided into: Accordingly impervious ($K_p < 0.1$ m/day), weakly permeable $0.1 \leqslant K_p \leqslant 10$, moderately permeable $10 \leqslant K_p \leqslant 500$ and highly permeable $K_p > 1000$ m/day.

The permeability of a rock is its property to allow the flow of fluids under certain pressure. Gap or clearance (n_0) is determined by the ratio between the area of the filtration channels to the total area of the filter and characterizes the property of the rock to percolate gases and liquids.

Permeability depends mainly on the size of pores and their total proportion in the rock and on its shape. According to Davis (1954), and Chakraborty and Taylor (1968), permeability determines the volume of percolating fluid, the path of the flow, and the frictional forces hampering the movement. Generally, permeability rises with the increase in the porosity of the rocks, especially if the pores are interconnected and open; but departures from this principle are known. The average diameter (D) of the pores is of appreciable significance. It can be calculated by a simplified formula, derived from Poiseuille's equation, reviewed by Rzhevsky (1971) as follows:

$$D = \sqrt{32 K_{per}/m_i} \quad \text{.......} \quad (3)$$

According to Leubenzon (1953), the porosity and permeability of rock is related by the following general expression:

$$K_1 = f(m_1, d_\alpha, \lambda) \quad \text{.......} \quad (4)$$

Nomenclature

K_{pr} or K_p	— coefficient of permeability
Q	— amount of water passing through section(s) in unit time
V_p	— Q/S, velocity or rate of percolation
$\triangle F$	— pressure drop during percolation of the liquid L
η	— viscosity of the liquid in poises (dyne sec/cm^2)
D	— average diameter of the pores
γ_w	— specific gravity of water
(m or m_i)	— porosity
d_α	— parameter, which characterizes the dimension of the constituent particles

λ — parameter, which studies the shape of the filtration channel
d_e — effective diameter of the particle
SL — f(m. ε)
ε — coefficient which studies the form and the state of the surface of the particles in the rock
r — dimension of the particle before deformation
R — radius of the spherical surface
θ — angle of packing
r_0 — radius of the assumed sphere of the solids
V_m — volume of a sphere of assumed radius r_0, (before deformation)
m_0 — porosity of the rock before deformation
C — constant equals to (r_0/r)
$r_{min.}$ — minimum dimension of the particle after volumetric deformation
n_0 — gap of clearance
S_m — area occupied by the particles in unit cross section
S — the area of the exposed surfaces of the cubical particles
u — linear change in the length of the side of an elementary cube
V_d — volume of the deformed rock
S_d — surface area of an elementary cube after deformation
V_a — average velocity of flow of liquids and gases
ν — dynamic viscosity of liquids and gases
dp/dl — pressure drop along the length of the pore
w — volumetric velocity of filtration = n. v_a
n — number of exposed surfaces of the cube
d_β — diameter of the particle = 2 r
dv/dx — rate of change of the velocity along the width of the channel

Mathematical Modelling for the Filtration Properties of Rocks

There are two general methods for determining porosity and permeability of the rocks, namely; experimental and analytical (mathematical modelling). The analytical method comprises the following elements:

i) Description of the geometrical structure of the filtration model.
ii) Definition of the geometrical parameters (porosity, gap or clearance).
iii) Deduction of the basic formulae for determining the coefficients of filtration and permeability of the rocks.

The mathematical methods give the possibility to take into consideration the following parameters:

i) Shape and form of the filtration channels and the regime of the movement of fluids in the percolating medium.
ii) Study the effect of the rock deformation around the mining opening (depth, dimension and shape of the opening, thickness of the ore body, angle of dip, physico-mechanical properties of rocks and ores).

Hence, only the analytical methods give a possibility to get relations between rock displacement and either porosity or the permeability of the rock.

The following general relationship is recommended for determining the permeability coefficient of the rock:

$$K = de^2 SL \qquad (5)$$

where

$$SL = F(m, \varepsilon).$$

Model of a Rock Composed of Deformed Spheres

In this model, it is assumed that the rock consists of deformed spheres. The surface of each deformed sphere consists a part of the side of the element shown in Fig. 1. The centers of the eight tangential spheres are disposed at the vertices of a rhombohedron.

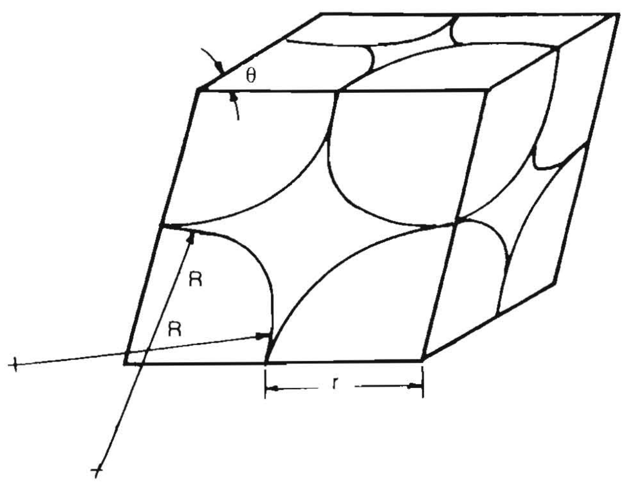

Fig. 1. Model of rock composed of deformed spheres.

As shown in Fig. 1, the geometrical parameters of the model are: The dimension of the particle (r), the radius of the spherical surface (R), and the angle of packing (θ) which equals 60 or 90°.

Porosity and Clearance befor Deformation

The volume of the initial cube (for $\theta = 90°$) equals to:

$$V = (2r)^3 = 8r^3$$

If V_m is the volume of solids which assumed to be a sphere of radius r_0, then

$$V_m = (4/3)\ \pi\ r_0^3$$

and the porosity before deformation will equal to:

$$m_0 = (V-V_m)/V = 1-(V_m/V) = 1-(4/3\,\pi\, r_0^3)/8r^3 = 1-(\pi/6)(r_0/r)^3. \qquad (6)$$

$$m_0 = 1 - 0.524C^3 \ldots\ldots\ldots\ldots \qquad (7)$$

where $C = r_0/r$, coefficient of the change of the particle shape.

If the rock consists of spherical particles, $C = 1$.

Under volumetric compressive deformation of the rock, the shape of constituent particles will be approximately a cube of volume equal to a volume of a sphere of radius r_0^3, hence

$$4/3\ \pi\ r_0^3 = 8\ r_{min}^3.$$

From this expression we get:

$$C_{max} = (r/_0\text{-}r_{min}) = \sqrt[3]{6/\pi} = 1.24 \ldots\ldots\ldots\ldots \qquad (8)$$

hence:-

$$1.0 < C < 1.24$$

$C = 1$ for spherical particles and 1.24 for cubical particles.

Consider a rock which consists of cubical particles as shown in Fig. 2; the gap or clearance is determined by the following formula:

$$n_0 = 1 - (S_m/S) \ldots\ldots\ldots\ldots \qquad (9)$$

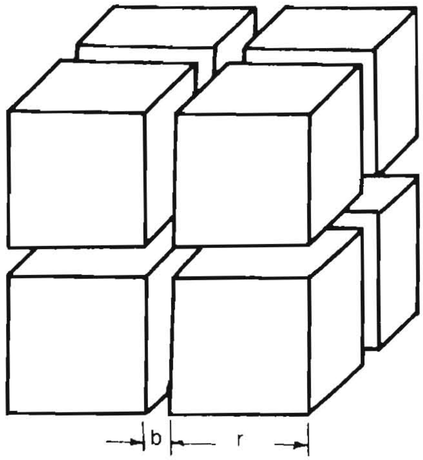

Fig. 2. Model of rock consisting of cubical grains with intermediate channels.

In this case we notice that $S = 4r^2$ and $S_m = 0.83\ \pi\ r_0^2$ for rocks formed of particles of cubic form.

The relation between the change in the cross-sectional area of the particle and its deformation can be expressed after small approximation as follows:

$$S_m = 1.71\ \pi\ r_0^2\ (1 - 0.415C) \quad \text{.......} \quad (10)$$

from equations (9) and (10) we get:

$$n_0 = 1 - \{(1.71\ \pi\ r_0^2\ (1-0.415C)/4r^2\} = 1-1.34C^2\ (1-0.415C)$$

$$= 1 - 1.34C^2 + 0.56C^3 \quad \text{.......} \quad (11)$$

Expressing the value of C in terms of porosity (equation 7) we get the following expression:

$$C = \sqrt[3]{(1 - m_0)/0.524} = (1/0.524)^{1/3} \quad \sqrt[3]{1 - m_0} =$$

$$= 1.24 \quad \sqrt[3]{(1 - m_0)} \quad \text{.......} \quad (12)$$

Equation (11) can be rewritten in the following form:

$$n_0 = 1 - 2.07 \sqrt[3]{(1 - m_0)^2} + 1.07\,(1 - m_0) \quad \ldots\ldots\ldots\ldots (13)$$

from equation (13) we get an approximate value of clearance in terms of rock porosity as:

$$n_0 \simeq 2/3\, m_0 \quad \ldots\ldots\ldots\ldots (14)$$

Porosity and Clearance for a Deformed Rock

Assume that the volume of the solid particles is not altered by deformation. For volumetric tension, the radius of the sphere R (Fig. 1) will decrease, while, for volumetric compression, this dimension will increase, and the form of the particles in the final stage of deformation will be approximated to a cube.

If before deformation, the length of the side of the elementary cube equals 2r, then after deformation this side will equal to $2(r \pm u)$. Hence, the volume of this cube after deformation will equal to $8(r \pm u)^3$.

As it was assumed, the volume of the rock particles are unchanged before and after deformation, hence the porosity of a deformed rock will be:

$$\begin{aligned} m &= 1 - (V_m / V) \\ &= 1 - \{(4/3)\ \pi\ r_0^3 / 8r^3\ [1 \pm (U/r)]^3\} \\ &= 1 - \{(\pi/6)C^3 / [1 \pm (U/r)]^3\} \quad \ldots\ldots\ldots\ldots (15) \end{aligned}$$

(U/r) in equation (15) will be expressed as (du/dx), rate of change in the length in one direction only. For triaxial deformation this relation is expressed as follows:

$$m = 1 - \{(1 - m_0) / [1 \pm (du/dx)]^3\} \quad \ldots\ldots\ldots\ldots (16)$$

The clearance of the deformed rock equals to:

$$m = 1 - (S_m / S_d) \quad \ldots\ldots\ldots\ldots (17)$$

S_m is determined from equation (10)

$$S_d = 4\,(r \pm u)^2 = 4\,r^2\,[1 + (du / dx)^2]$$

substituting the value of n_0 from equation (14) we get:

$$n = 1 - \{(1 - 2/3\, m_0) / [1 + (du / dx)^2] \quad \ldots\ldots\ldots\ldots (18)$$

Coefficient of Permeability for a Deformed Rock

The pores within the structure of the rock can be considered either circular cylinders ($C = 1.0$) or slotted channels ($C = 1.24$). The problem is summarized in the determination of the coefficient of permeability for the above stated considerations.

a) Coefficient of permeability for pores of the shape of circular cylinders

According to Sherepan (1957) and Ermekov (1967), the filtration of gasses and liquids in mining practice is usually illustrated by a laminar flow. The average velocity of flow of liquids and gases in a circular cylinder assuming laminar type of flow is determined from the following relation:

$$V_a = (S/8\,\pi\,\mu).\,(dp/dl) \qquad \text{.. (19)}$$

where:

S – cross-sectional area of the rock = nL^2,

L – length of the side of the cube = $2r$.

Hence the average velocity V_a can be rewritten in the following form:

$$V_a = (nL^2/8\,\pi\,\mu) \times (dp/dl) \qquad \text{.. (20)}$$

The volumetric velocity of filtration of fluids in pores, W, will be given by:

$W = n\ V_a$ *i.e.*

$$W = (n^2L^2/8\,\pi\,\mu).(dp/dl) \qquad \text{.. (21)}$$

By Darcy's law

$$W = (K_p/\mu)\ (dp/dl) \qquad \text{.. (22)}$$

Comparing equations (21) and (22) we get:

$$K_p = n^2L^2\ /\ 8\,\pi \qquad \text{.. (23)}$$

For a deformed cube, the length of the side will equal to:

$$L = (2r \pm U) = 2r\,[1 \pm (du/dx)]$$

From equation (18) we get:

$$n = \{[1 \pm (du/dx)]^2 - (1-2/3\ m_0)\} \ / \ [1+(du/dx)]^2$$

$$= \{1 \pm 2du/dx \pm (du/dx)^2 - 1+2/3\ m_0\} \ / \ [1+(du/dx)]^2$$

$$= \{2/3\ m_0 \pm 2(du/dx) + (du/dx)^2\} \ / \ [1 + (du/dx)]^2 \quad \dots\dots \quad (24)$$

Substitute the value of n and L in equation (23) we get:

$$K_p = (4r^2/8\ \pi)\ [1+(du/dx)]^2\ \{2/3\ m_0 \pm 2(du/dx)+(du/dx)^2/\ [1 + (du/dx)]^2\} \text{ or}$$

$$K_p = r^2/2\,\pi\ \{2/3\ m_0 \pm 2(du/dx) + (du/dx)^2 \ / \ [1+(du/dx)]^2\} \quad \dots\dots \quad (25)$$

(du/dx) usually equals (0.1−0.2), hence $(du/dx)^2$ can be neglected in equation (25), and this equation is reduced to the following form:

$$K_p = (d_\beta^2/8\,\pi)\ \{[2/3\ m_0 \pm 2(du/dx)] \ / \ [1 + (du/dx)]^2\} \quad \dots\dots \quad (26)$$

In practice, d_β is measured in mms, and the coefficient of permeability in Darcys (1 Darcy = 1.02×10^{-8} cm^2), equation (26) can be rewritten as:

$$K_p = 3.94\ d_\beta^2 \times 10^4\ \{[2/3\ m_0 \pm 2(du/dx)] \ / \ [1+(du/dx)]^2\} \quad \dots\dots \quad (27)$$

From equation (27), the coefficient of permeability before deformation (du/dx) = 0.

$$K_0 = 1.75 \times 10^4\ d_\beta^2\ m_0^2 \quad \dots\dots \quad (28)$$

b) Coefficient of permeability for the pores having the shape of slotted channels

In the horizontal position of the channel, the tractive force F (Fig. 3) resulting from the pressure difference dp equals to:

$$F = 2\ .\ x.\ L.\ dp \quad \dots\dots \quad (29)$$

where:

2x = thickness of the layer;

L = length of the slotted channel.

The force F resisting the tangential force T, will equal to:

$$-\ T = 2L\ dl\ \mu\ (dv/dx). \quad \dots\dots \quad (30)$$

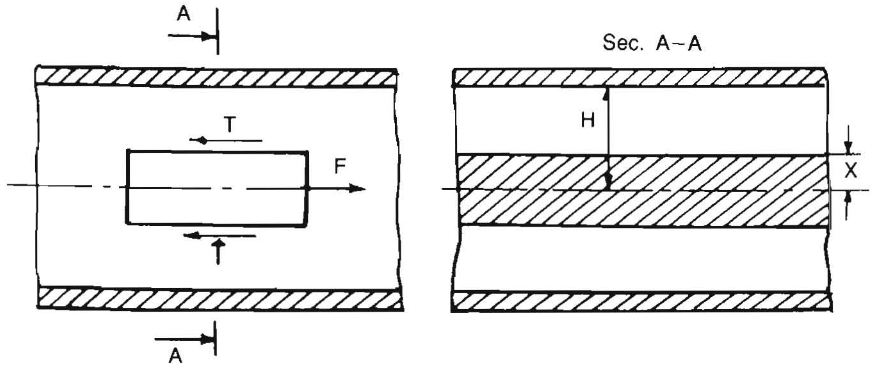

Fig. 3. Scheme for determining the average velocity of the flow for fluids in a slotted channel.

At the equilibrium position we have (F = T):

$2xL.dp = -2L\ dl\ \mu\ (dv/dx)$.

$dv/dx = (-1/\mu)(dp/dl).x$ & $dv = (-1/\mu)(dp/dl)\ .\ x\ dx$

for (dp/dl) = constant, we get:

$$V = -(1/\mu)(dp/dl)(H^2 - Y^2) \qquad (31)$$

and the average velocity of flow equals to:

$$V_a = (1/H)\int_O^H V.dx = H^2/2\,\mu)(dp/dl) \qquad (32)$$

Taking into consideration that the cross-sectional area of one pore equals to:

$$S = 2HL \text{ we get } H = (S/2L)$$

$$V_a = (S^2/8\,\mu L^2)(dp/dl) \qquad (33)$$

The cross-sectional area of one channel is expressed can be terms of clearance (n) as $S = 1/2\ nL^2$. Substitute this value in equation (33) we get an expression for the volumetric velocity of filtration as follows:

$$W = V_a.\ n = (L^2n^3/32\,\mu)(dp/dl) \qquad (34)$$

The coefficient of permeability is expressed in the following form:

$$K_p = \mu\ W/(dp/dl) = L^2n^3/32 \qquad (35)$$

Substitute the values of n from equation (24) and $L = 2r\,[1 \pm (du/dx)]$ and summarizing we get:

$$Kp = \{(4/32)r^2\,[1+(du/dx)]^2.\ [2/3\ m_0 \pm 2(du/dx)+(du/dx)^2]^3\}\ /\ [1+(du/dx)]^6$$

$$K_p = d_\beta^2/32\,[2/3\ m_0 \pm 2(du/dx)^3]\,/\,[1+(du/dx)]^4 \qquad \text{.............................. (36)}$$

for d in mms and K_p in Darcys we get:

$$K_p = 3.2\ d_\beta^2\ .10^4[2/3\ m_0 \pm 2(du/dx)^3]\ /\ [1+(du/dx)]^4$$

$$K_0 = 3.2\,d_\beta^2 \times 10^4\,(2/3\ m_0)^3 = 0.92.10^4\ d_\beta^2\,m_0^3 \qquad \text{................................. (37)}$$

From equations (27) and (36) we can deduce that the coefficient of permeability depends on the dimensions of the particles which compose the material of the rock, the deformation (du/dx) and the porosity of the rock before deformation m_0.

Discussion

It is proved in equation (14) that the clearance of undeformed rocks n_0 equals two thirds of its porosity m_0. The porosity of the undeformed rocks determines the relative volume of pores, while the clearance of the rock (n_0) is related to the surface area as expressed by equations (6 & 9). The clearance of deformed rocks is mathematically expressed by equation (18).

The coefficient of permeability K_p for undeformed rocks containing cylindrical shaped pores can be calculated by making use of equation (27). For undeformed rocks equation (28) is applied.

If the pores are slotted shaped channels, the coefficient of permeability can be evaluated for deformed rocks by the application of equation (36), while equation (37) is used for undeformed rocks.

The above mentioned formulae are recommended for both experimental and field works.

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استنتاج العلاقات الرياضية لدراسة تأثير تشكل الصخور على مساميتها ونفاذيتها

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نوقشت الطرق المختلفة لإيجاد النماذج الرياضية لدراسة خواص النفاذية في الصخور، واستنتاج العلاقات الرياضية التي تربط بين مسامية الصخور وقابليتها لنفاذ الغازات والسوائل في حالتين:
عندما يكون الصخر في حالته العادية أو عندما يقع تحت أحمال خارجية. وقد تم في هذا البحث حساب معامل النفاذ الحجمي، والتعبير عنه بصورة رياضية في حالتين: الحالة الأولى تفترض أن شكل الفراغ الداخلي على هيئة إسطوانة دائرية، وفي الحالة الثانية افترض شكل الفراغ على هيئة المجرى.