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Fuzzy Separation Axioms and Fuzzy Continuity

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ABSTRACT. Let (X, τ) be a topological space and let $\omega(\tau)$ be the set of all lower semicontinuous functions defined from X into the closed unit interval, [0,1]. In this paper we define two fuzzy separation axioms, namely; functionally Hausdorff and complete regularity. Then we prove a) the space (X, τ) is functionally Hausdorff if and only if the fuzzy space $(X, \omega(\tau))$ is functionally Hausdorff, and b) the space (X, τ) is completely regular if and only if the fuzzy space $(X, \omega(\tau))$ is completely regular.

1. Introduction

After Zadeh created fuzzy sets in his classical paper; Zadeh (1965) and Chang (1968) used them to introduce the concept of a fuzzy topology. The concept of induced fuzzy topological spaces was introduced by Weiss (1975). Since then several authors continued the investigation of such spaces. However, most attention is paid to the extension of the separation notions to fuzzy topological spaces.

2. Preliminaries

Let X be a nonempty set. A fuzzy set λ on X is a function from X into the closed unit interval [0,1]. As usual 0 and 1 denote the fuzzy sets given by $0(x)=0$ and $1(x)=1$, $x \varepsilon X$. We write $\lambda \subseteq \mu$ if $\lambda(x) \leq \mu(x)$ for all $x \epsilon X$. By $\lambda=\mu$ we mean $\lambda \subseteq \mu$ and $\mu \subseteq \lambda$. If $\{\lambda_i,\ i \varepsilon I\}$ is a collection of fuzzy sets on X, then we define

$$(\cup\lambda_i)(x) = \sup\ \{\lambda_i(x): i \varepsilon I\},\ x \varepsilon X, \text{ and}$$
$$(\cap\lambda_i)(x) = \inf\ \{\lambda_i(x): i \varepsilon I\},\ x \varepsilon X.$$

The complement λ' of a fuzzy set λ on X is given by $\lambda'(x) = 1\text{-}\lambda(x)$, $x\varepsilon X$. If $A \subseteq X$, then χ_A denotes the characteristic function of A.

2.1 *Definition* (Wong 1974)

A fuzzy point p in a set X is a fuzzy set in X given by

$$p(x) = t \text{ for } x = x_p \ (0 < t < 1), \text{ and}$$
$$p(x) = 0 \text{ for } x \neq x_p.$$

The support of a fuzzy set λ in X is the ordinary subset supp $\lambda = \lambda^{-1}(0,1] \subseteq X$. The support of a fuzzy point p is often written as x_p and its value $p(x_p)\ \varepsilon(0,1)$. Two fuzzy points p and q in X are said to be distinct iff their supports are distinct. A fuzzy point p in X is said to belong to a fuzzy set λ in X (notation: $p\varepsilon\lambda$) iff $p(x_p) < \lambda(x_p)$.

2.2 *Definition* (Wong 1974)

A fuzzy topology on a set X is a collection T of fuzzy sets in X such that

(i) $0,1\ \varepsilon T$
(ii) If $\lambda,\mu\varepsilon T$ then $\lambda\cap\mu\varepsilon T$;
(iii) If $\lambda_i\varepsilon T$, $i\varepsilon I$, then $\cup\lambda_i\varepsilon T$.

Members of T are called T-open fuzzy sets (or simply open sets) and the pair (X, T) is called a fuzzy topological space, in short, an fts. Complements of open fuzzy sets are called closed fuzzy sets. When X carries an ordinary topology τ, there is a simple fuzzy topology on X associated with τ, namely the class of all lower semicontinuous functions between (X,τ) and $([0,1], \tau_u)$ with τ_u the usual topology on $[0,1]$. This fuzzy topology is denoted by $\omega(\tau)$ and is called by F. Conrad (1980) the natural fuzzy topology on X associated with the ordinary topology τ. The closure $cl\,\lambda$ (or $\bar{\lambda}$) and the interior int λ (or λ^o) of a fuzzy set λ in a fts (X, T) are defined by

$$\bar{\lambda} = \cap \{\mu : \mu \text{ is a closed fuzzy set and } \lambda \subseteq \mu\}$$
$$\lambda^o = \cup \{\mu : \mu \text{ is an open fuzzy set and } \mu \subseteq \lambda\}$$

2.3 *Definition*

Consider two ordinary sets X and Y; a fuzzy set λ in X, a fuzzy set μ in Y, and a function f from X into Y. The direct image of λ under f is the fuzzy set $f(\lambda)$ in Y given by

$$f(\lambda)(y) = \sup \{\lambda(x) : x\varepsilon f^{-1}(\{y\})\} \text{ if } f^{-1}(\{y\}) \neq \phi$$
$$= 0 \qquad \text{if } f^{-1}(\{y\}) = \phi$$

The inverse image of μ under f is the fuzzy set $f^{-1}(\mu)$ in X given by

$$f^{-1}(\mu)(x) = \mu(f(x)) \ , \ x \varepsilon X.$$

2.4 *Definition*

Let f: $(X, T_1) \rightarrow (Y, T_2)$ be a function from a fts (X, T_1) to a fts (Y, T_2). The function f is a fuzzy continuous iff the inverse image of every T_2-open fuzzy set in Y is T_1-open or, equivalently, iff the inverse image of every T_2-closed fuzzy set in Y is T_1-closed.

3. Fuzzy Separation Axioms and the Space $\omega(\tau)$

Several authors introduced different definitions of separation properties for fuzzy topologies (see, *e.g.*, Hutton 1975 and 1977, Hutton and Reilly 1980, Ming and Ming 1980 and Srivastava, *et al.* 1981). The Hausdorff axiom has had a hard life in fuzzy set theory since many authors proposed different definitions (*e.g.*, Ming and Ming 1980, Sarkar 1981 and Wong 1974).

Let us state the following two results before introducing our definitions of separation axioms.

3.1 *Theorem*

Let (X, T) be an fts and let λ be a fuzzy set in X. Then we have

i) $\bar{\lambda}$ is the smallest closed fuzzy set containing λ;
ii) λ^0 is the largest open fuzzy set contained in λ;
iii) $\lambda^0 = \lambda^{1-1}$

For any $r \varepsilon (0,1)$, $A \subseteq X$, let $r\chi_A$ be defined by $(r\chi_A)(x) = r$ if $x \varepsilon A$ and $(r\chi_A)(x) = 0$ if $x \varepsilon X - A$.

3.2 *Theorem*

Let $(X, \omega(\tau))$ be the fuzzy space associated with the ordinary topology τ on X, and let λ be any fuzzy set in X, $A \subseteq X$ and $r \varepsilon (0,1)$. Then we have:

i) $cl\ (\lambda^{-1}(a,1]) \subseteq (cl\lambda)^{-1}\ [a,1]$ for all $a\varepsilon\ [0,1]$.
ii) $cl\ \chi_A = \chi_{\bar{A}}$
iii) $cl\ (r\ \chi_A) = r\ \chi_{\bar{A}}$
iv) $cl\ \lambda \subseteq \chi_{cl(\mathrm{supp}\ \lambda)}$.
v) $r\ \chi_A\ \varepsilon\omega(\tau))$ if and only if $A\varepsilon\tau$.

3.3 *Definition*

A fuzzy topological space (X,T) is said to be

i) T_0 iff for any two distinct fuzzy points p,q in X, there exists an open fuzzy set μ such that ($p\varepsilon\mu$ and $\mu \cap q=0$) or ($q\varepsilon\mu$ and $\mu\cap p=0$).

ii) $T_{0\omega}$ *iff for any two distinct fuzzy points* p,q in X, there exists an open fuzzy set μ such that $p\varepsilon\mu\subseteq q'$ or $q\varepsilon\mu\subseteq p'$.

iii) T_1 iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1$, $\mu_1 \cap q = 0, q\varepsilon\mu_2$ and $\mu_2 \cap p = 0$.

iv) $T_{1\omega}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1\subseteq q'$ and $q\varepsilon\mu_2\subseteq p'$.

3.4 *Theorem*

Let (X,τ) be a topological space. Then the following statements are equivalent:

i) (X,τ) is a T_0-space.
ii) $(X,\omega(\tau))$ is a T_0-space.
iii) $(X,\omega(\tau))$ is a $T_{0\omega}$-space.

3.5 *Theorem*

Let (X,τ) be a topological space. Then the following statements are equivalent:

i) (X,τ) is a T_1-space.
ii) $(X,\omega(\tau))$ is a T_1-space.
iii) $(X,\omega(\tau))$ is a $T_{1\omega}$-space.

Proof. (i) $\rightarrow$ (ii) Let (X,τ) be a T_1-space and let p,q be two distinct fuzzy points in X. Then $x_p \neq x_q$. Thus there exist $U, V_\varepsilon\tau$ such that $x_p\varepsilon U, x_p\notin V$, $x_q\notin U$ and $x_q\varepsilon V$. It is clear that $p\varepsilon\chi_U, q\varepsilon\chi_V, \chi_U\varepsilon\omega(\tau)$, $\chi_V\varepsilon\omega(\tau)$ and $\chi_U \cap q = \chi_V \cap p = 0$.

(ii) $\rightarrow$ (iii) follows from the definition.

(iii) $\rightarrow$ (i) Let $(X,\omega(\tau))$ be a $T_{1\omega}$-space and let x_0, y_0 be two distinct elements in X. Take p,q to be the fuzzy points in X for which $p(x_0) = q(y_0) = 0.9$. Then p,q are distinct fuzzy points. Thus there exist λ_1, $\lambda_2\varepsilon\omega$ (τ) such that $p\varepsilon\lambda_1 \subseteq q'$ and $q\varepsilon\lambda_2 \subseteq p'$. It is clear that $U=\lambda_1^{-1}(0.9,1]$ $\varepsilon\tau$, $V = \lambda_2^{-1}$ $(0.9,1]$ $\varepsilon\tau$, $x_0\varepsilon U, y_0\varepsilon V$, $x_0 \notin V$ and $y_0\notin U$.

3.6 *Definition*

A fuzzy topological space, (X,T) is said to be

i) T_2 iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1

and μ_2 such that $p\varepsilon\mu_1$, $q\varepsilon\mu_2$ and $\mu_1 \cap \mu_2 = 0$.

ii) $T_{2\omega}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1$, $q\varepsilon\mu_2$ and $\mu_1 \subseteq \mu_2'$.

iii) $T_{2\frac{1}{2}}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1$, $q\varepsilon\mu_2$ and $\overline{\mu_1} \cap \overline{\mu_2} = 0$.

iv) $T_{2\frac{1}{2}\omega}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1$, $q\varepsilon\mu_2$ and $\overline{\mu_1} \subseteq \overline{\mu_2}^{-1}$.

3.7 *Theorem*

Let (X,τ) be a topological space. Then the following statements are equivalent:

i) (X,τ) is a T_2-space.
ii) $(X,\omega(\tau))$ is a T_2-space.
iii) $(X,\omega(\tau))$ is a $T_{2\omega}$-space.

3.8 *Theorem*

Let (X,τ) be a topological space. Then the following statements are equivalent:

i) (X,τ) is a $T_{2\frac{1}{2}}$-space.
ii) $(X,\omega(\tau))$ is a $T_{2\frac{1}{2}}$-space.
iii) $(X,\omega(\tau))$ is a $T_{2\frac{1}{2}\omega}$-space.

Proof. (i) $\rightarrow$ (ii) Let (X,τ) be a $T_{2\frac{1}{2}}$-space and let p,q be two distinct fuzzy points in X. Then $x_p \neq x_q$. Thus there exist $U, V\varepsilon\tau$ such that $x_p\varepsilon\ U, x_q\varepsilon V$ and $\overline{U} \cap \overline{V} = \phi$. By *Theorem 3.2* (ii), $\overline{\chi_U} \cap \overline{\chi_V} = \chi_{\overline{U}} \cap \chi_{\overline{V}} = \chi_{\overline{U}\cap\overline{V}} = \chi_\phi = 0$. It is now clear that $p\varepsilon\chi_U$, $q\varepsilon\chi_V$, $\chi_U\varepsilon\omega(\tau)$, $\chi_V\varepsilon\omega(\tau)$ and $\overline{\chi_U} \cap \overline{\chi_V} = 0$.

(ii) $\rightarrow$ (iii) is immediate.

(iii) $\rightarrow$ (i) Let $(X\omega(\tau))$ be a $T_{2\frac{1}{2}\omega}$-space and let x_0,y_0 be two distinct elements in X. Take p,q to be the fuzzy points in X for which $p(x_0) = q(y_0) = 0.9$. Then there exist λ_1, $\lambda_2\varepsilon\omega(\tau)$ such that $p\varepsilon\lambda_1$, $q\varepsilon\lambda_2$, and $\overline{\lambda_1}\subseteq\overline{\lambda_2}^{-1}$. It is clear that $U = \lambda_1^{-1}(0.9,1]\varepsilon\tau$, $V = \lambda_2^{-1}(0.9,1]\varepsilon\tau$, $x_0\varepsilon U$, $y_0\varepsilon V$ and $\overline{U}\cap\overline{V} = \phi$. Indeed, if $x\varepsilon\ \overline{U}\cap\overline{V}$, then according to Theorem 3.2(i), $\overline{\lambda_1}(x) \geqslant 0.9$ and $\overline{\lambda_2}(x) \geqslant 0.9$ which contradicts the fact $\overline{\lambda_1}\subseteq \overline{\lambda_2}^{-1}$. This completes the proof of our theorem.

3.9 *Definition*

A fuzzy topological space (X,T) is said to be regular iff for every fuzzy point p in X and every closed fuzzy set λ in X such that $p\varepsilon\lambda'$, there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1$, $\lambda\subseteq\mu_2$ and $\mu_1\subseteq\mu_2'$.

An important and useful characterization of fuzzy regularity is given in the following theorem.

3.10 *Theorem*

Let (X,T) be an fts. Then the following statements are equivalent:

i) (X,T) is regular.
ii) For every fuzzy point p in X and for every open fuzzy set λ such that $p\varepsilon\lambda$, there exists an open fuzzy set μ such that $p\varepsilon\bar{\mu}\subseteq\bar{\mu}\subseteq\lambda$.
iii) For every closed fuzzy set λ in X and any fuzzy point $p\varepsilon\lambda'$, there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1$, $\lambda\subseteq\mu_2$ and $\bar{\mu}_1\subseteq\bar{\mu}_2'$.

Proof: (i) → (ii). Let p be any fuzzy point in X and let λ be an open fuzzy set in X such that $p\varepsilon\lambda$. Since λ' is closed and $p\varepsilon\lambda''$, therefore; by the regularity of X; there exist open fuzzy sets σ_1 and σ_2 such that $p\varepsilon\sigma_1$, $\lambda'\subseteq\sigma_2$ and $\sigma_1\subseteq\sigma_2'$. Since σ_2' is closed, therefore $\bar{\sigma}_1\subseteq\sigma_2'^{-} = \sigma_2'\subseteq\lambda$. Consequently $p\varepsilon\sigma_1 \subseteq \bar{\sigma}_1 \subseteq \lambda$.

(ii) → (iii). Let λ be a closed fuzzy set in X and let p be any fuzzy point in X such that $p\varepsilon\lambda'$. Using (ii), there exists an open fuzzy set σ_1 such that $p\varepsilon\sigma_1 \subseteq \bar{\sigma}_1 \subseteq \lambda'$. Applying (ii) again, there exists an open fuzzy set σ_2 such that $p\varepsilon\sigma_2 \subseteq \bar{\sigma}_2 \subseteq \sigma_1$. The proof is completed by taking $\mu_1 = \sigma_2$ and $\mu_2 = \sigma_1^{-/}$ because $p\varepsilon\mu_1$, $\lambda \subseteq \mu_2$ and $\bar{\sigma}_2 \subseteq \sigma_1^{-/-/}$. Indeed $\sigma_1\subseteq\bar{\sigma}_1$ implies $\sigma_1^0 \subseteq \sigma_1^{-0}$. Using Theorem 3.1 (iii), we get $\sigma_1 \subseteq \sigma_1^{-/-/}$. But since $\bar{\sigma}_2 \subseteq \sigma_1$, so we get $\bar{\sigma}_2 \subseteq \sigma_1^{-/-/}$.

(iii) → (i). Let λ be a closed fuzzy set and let p be a fuzzy point in X such that $p\varepsilon\lambda'$. Using (iii), there exist open fuzzy sets μ_1 and μ_2 such that $p\varepsilon\mu_1$, $\lambda \subseteq \mu_2$ and $\bar{\mu}_1 \subseteq \bar{\mu}_2'$. Since $\mu_1 \subseteq \bar{\mu}_1$ and $\bar{\mu}_2' \subseteq \mu_2'$, therefore $\mu_1 \subseteq \mu_2'$. Hence (X,T) is regular.

3.11 *Theorem*

Let (X,τ) be a topological space. Then, (X,τ) is regular if and only if $(X,\omega(\tau))$ is regular.

Proof: Let (X,τ) be a regular space. Let p be a fuzzy point in X and $\lambda\varepsilon\omega(\tau)$ such that $p\varepsilon\lambda$. Let $r = \frac{1}{2}\,(p(x_p) + \lambda(x_p))$. Then $p(x_p) < r < \lambda(x_p)$, *i.e.* $x_p\varepsilon\lambda^{-1}(r,1]$. Using the regularity of X, there exists $U\varepsilon\tau$ such that $x_p\varepsilon\ U \subseteq \bar{U} \subseteq \lambda^{-1}(r,1]$. Using Theorem 3.2 (iii), the following inclusions become clear: $p\varepsilon r\ \chi_U \subseteq r\ \chi_{\bar{U}} \subseteq r\ \chi_{\lambda^{-1}(r,1]} \subseteq \lambda$.

The fact that $r\chi_{\bar{U}}\varepsilon\omega(\tau)$ completes the proof of the first implication.

Conversely, let $(X,\omega(\tau))$ be a regular space. Let $U\varepsilon\tau$ and $x_0\varepsilon U$. Take p to be the fuzzy point in X for which $p(x_0) = 0.9$. Then $p\varepsilon\ \chi_U$ and $\chi_U\varepsilon\omega(\tau)$. Therefore, there exists $\mu\varepsilon\omega(\tau)$ such that $p\varepsilon\mu \subseteq \bar{\mu}\subseteq\chi_U$. Using Theorem 3.2 (i), we get $cl(\mu^{-1}$

$(0.9,1]) \subseteq (cl\mu)^{-1}[0.9,1] \subseteq \chi_U{}^{-1}[0.9,1] = U$. Let $V=\mu^{-1}(0.9,1]$. Then $V\varepsilon\tau$ and $x_0 \varepsilon V \subseteq \bar{V} \subseteq U$. This completes the proof of our theorem.

3.12 *Definition*

A fuzzy topological space (X,T) is said to be normal iff for every pair of closed fuzzy sets λ_1, λ_2 such that $\lambda_1 \subseteq \lambda_2'$, there exist open fuzzy sets μ_1, μ_2 such that $\lambda_1 \subseteq \mu_1 \subseteq \mu_2' \subseteq \lambda_2'$.

An important and useful characterization of fuzzy normality is given in the following theorem.

3.13 *Theorem*

Let (X,T) be an fts. Then the following statements are equivalent:

i) (X,T) is normal.

ii) For every closed fuzzy set σ in X and for every open fuzzy set λ such that $\sigma \subseteq \lambda$, there exists an open fuzzy set μ such that $\sigma \subseteq \mu \subseteq \bar{\mu} \subseteq \lambda$.

iii) For every closed fuzzy set σ in X and any closed fuzzy set $\lambda \subseteq \sigma'$, there exist open fuzzy sets μ_1 and μ_2 such that $\lambda \subseteq \mu_1$, $\sigma \subseteq \mu_2$ and $\bar{\mu}_1 \subseteq \bar{\mu}_2'$.

iv) For every closed fuzzy set λ and open fuzzy set μ such that $\lambda \subseteq \mu$, there exists a fuzzy set σ such that $\lambda \subseteq \sigma^0 \subseteq \bar{\sigma} \subseteq \mu$.

Proof: For the proof of the equivalence of (i), (ii) and (iii), imitate the proof of Theorem 3.10.

(ii)→(iv) is immediate.

(iv)→(ii) Let σ be a closed fuzzy set in X and let λ be any open fuzzy set such that $\sigma \subseteq \lambda$. Using (iv), there exists a fuzzy set δ such that $\sigma \subseteq \delta^0 \subseteq \bar{\delta} \subseteq \lambda$. The proof is completed by taking $\mu = \delta^0$. Indeed $\mu = \delta^0 \subseteq \delta$ implies $\bar{\mu} \subseteq \bar{\delta}$. Consequently, μ is an open fuzzy set in X satisfying $\sigma \subseteq \mu \subseteq \bar{\mu} \subseteq \lambda$.

3.14 *Theorem*

Let (X,τ) be a topological space. Then (X,τ) is normal if and only if $(X,\omega(\tau))$ is normal.

Proof: (→) Let λ be a closed fuzzy set and μ be any open fuzzy set in the fuzzy topological space $(X,\omega(\tau))$ such that $\lambda \subseteq \mu$. Then $\lambda: X \to [0,1] \subseteq R$ is an upper semicontinuous function and $\mu: X \to [0,1] \subseteq R$ is a lower semicontinuous function. By Tong (1952), there exists a continuous function $f: X \to R$ such that $\lambda(x) \leqslant f(x) \leqslant \mu(x)$ for every $x \varepsilon X$. Since $\lambda(x) \geqslant 0$ and $\mu(x) \leqslant 1$ for every $x \varepsilon X$, therefore $0 \leqslant f(x) \leqslant 1$ for all $x \varepsilon X$, *i.e.* f is actually a fuzzy set in X. Since $f: X \to [0,1]$ is continuous, so f is lower semicontinuous and upper semicontinuous. Hence $\bar{f} = f = f^{\circ}$. Thus, by Theorem 3.13 (iv), $(X,\omega(\tau))$ is normal.

($\leftarrow$) Conversely, let $(X,\omega(\tau))$ be normal. Let A,B be two disjoint closed sets in (X,τ). Then χ_A, χ_B are closed fuzzy sets in $(X,\omega(\tau))$ and $\chi_A \subseteq \chi'_B$. Thus there exist $\lambda,\mu\varepsilon\omega(\tau)$ such that $\chi_A \subseteq \lambda$, $\chi_B \subseteq \mu$ and $\lambda \subseteq \mu'$. Let $U=\lambda^{-1}$ (0.9,1] and $V=\mu^{-1}(0.9,1]$. Then $U,V\varepsilon\tau$, $A \subseteq U, B \subseteq V$ and $U \cap V = \phi$. This completes the proof of our theorem.

To proceed for our next results, we need the following definition.

3.15 *Definition*

A family β of fuzzy open sets in X is called a base for a fts (X,T) iff members of T can be written as unions of members of β.

3.16 *Definition*

A family ψ of fuzzy open sets in X is called a subbase for a fts (X,T) iff the family of finite intersections of members of ψ forms a base for (X,T).

3.17 *Definition* (Hutton 1975)

The fuzzy unit interval $[0,1](L)$ is the set of all monotonic decreasing functions λ: $R \to L$ satisfying:

i) $\lambda(t) = 1$ for $t < 0$, $t\varepsilon R$,
ii) $\lambda(t) = 0$ for $t > 1$, $t\varepsilon R$;

after the identification of λ: $R \to L$ and μ:$R \to L$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for every $t\varepsilon R$ (where $\lambda(t-)=\inf\{\lambda(s): s<t\}$ and $\lambda(t+)=\sup\{\lambda(s): s>t\}$).

We define an L-fuzzy topology on [0,1] (L) by taking as a subbase $\{L_t, R_t: t\varepsilon R\}$ where we define

$$L_t(\lambda) = (\lambda(t-))' \text{ and } R_t(\lambda) = \lambda(t+).$$

This topology is called the usual topology for [0,1] (L).

It is easy to see that $\beta = \{R_a \cap L_b: a,b\varepsilon R\}$ is indeed a base for the usual fuzzy unit interval [0,1] (L).

The following result is another characterization of normal space.

3.18 *Theorem* (Hutton 1975)

A fuzzy topological space (X,T) is normal iff for every closed set λ and open set μ such that $\lambda \subseteq \mu$, *there exists a continuous function* f: $(X,T) \to [0,1]\ (L)$ such that for every $x \varepsilon X$

$$\lambda(x) \leq f(x)\ (1-) \leq f(x)\ (0+) \leq \mu(x).$$

3.19 *Definition*

A fuzzy topological space (X,T) is called completely regular iff for every open fuzzy set λ and every fuzzy point p in X such that $p\varepsilon\lambda$, there exists a continuous function f:$(X,T) \to [0,1]\ (L)$ such that for every $x\varepsilon X$

$$p(x) \leq f(x)\ (1-) \leq f(x)\ (0+) \leq \lambda(x).$$

3.20 *Theorem*

A topological space (X,τ) is completely regular if and only if $(X,\omega(\tau))$ is completely regular.

Proof: Let (X,τ) be a completely regular space. Let $\lambda\varepsilon\omega(\tau)$ and p be a fuzzy point in X such that $p\varepsilon\lambda$. Let $r = \frac{1}{2}(p(x_p) + \lambda(x_p))$. Then $p(x_p) < r < \lambda(x_p)$, *i.e*, $x_p\varepsilon\lambda^{-1}(r,1]$. Since $\lambda^{-1}(r,1]\varepsilon\tau$, there exists a continuous function f: $(X,\tau) \to [0,1]$ such that $f(x_p) = 1$ and $f(X-\lambda^{-1}(r,1]) = 0$. Define g: $(X,\omega(\tau)) \to [0,1]\ (L)$ as follows:

$$\begin{aligned} g(x)(t) &= r\chi_{f^{-1}(t,1]}\ (x) \text{ if } t\geq 0,\ t\varepsilon R,\ x\varepsilon X; \\ &= 1 \qquad\qquad\quad \text{ if } t<0,\ t\varepsilon R,\ x\varepsilon X. \end{aligned}$$

Then $g(x)\varepsilon\ [0,1]\ (L)$ for every $x\varepsilon X$, *i.e.* g is a well-defined function. It is easy to observe that $g^{-1}(R_t) = r\chi_{f^{-1}(t,1]}\ \varepsilon\omega(\tau)$ if $t\geq 0$, and $g^{-1}(R_t) = 1\ \varepsilon\omega(\tau)$ if $t<0$. We can also observe that $g^{-1}(L_t) = r\chi_{f^{-1}[0,t)}\ \varepsilon\omega(\tau)$ if $t>0$, and $g^{-1}(L_t) = 0\ \varepsilon\omega(\tau)$ if $t\leq 0$. Hence g is a continuous function. Doing some calculations one can verify the following inequalities:

$$p(x)\leq g(x)\ (1-)\leq g(x)\ (0+)\leq \lambda(x),\ x\varepsilon X.$$

Conversely, let $(X,\omega(\tau))$ be a completely regular space. Let $x_0\varepsilon X$ and $U\varepsilon\tau$ such that $x_0\ \varepsilon U$. Take p to be the fuzzy point with $p(x_0) = 0.9$. Then $p\ \varepsilon\ \chi_U$ and $\chi_U\varepsilon\omega(\tau)$. Thus there exists a continuous function f: $(X,\ \omega(\tau)) \to [0,1]\ (L)$ such that $p(x)\leq f(x)(1-)\leq f(x)(0+)\leq \chi_U(x)$ is true for all $x\varepsilon X$. It follows that $f(x_0)(1-)\geq 0.9$ and $f(x)(0+) = 0$ for all $x\varepsilon X-U$. Define g: $(X,\ \tau) \to [0,1]$ by $g(x) = \inf\ \{t\varepsilon R$: $f(x)$

$(t-)<t\}$. Since $f(x)(2-) = 0<2$ and $f(x)(r-) = 1>r$ for all $r<0$, it follows that g is a well defined function. The fact that $f(x)(r-) = 0 < r$ for all $r > 1$ implies that $g(x) \leqslant 1$. Since $f(x_0)(0.9) \geqslant f(x_0)(1-) \geqslant 0.9$, we have $f(x_0)(t-) \geqslant f(x_0)(0.9-) \geqslant 0.9 \geqslant t$ whenever $t \leqslant 0.9$. It follows that $g(x_0) > 0.9$. Now, if $x \varepsilon X-U$ then $f(x)(t-) = 0<t$ whenever $t>0$. Hence $g(x) = 0$, *i.e.* $g(X-U) = 0$. Doing some calculations, we find that $g^{-1}[0,t) = (f^{-1}(L_t))^{-1}(1-t,1] \varepsilon \tau$ and

$$g^{-1}(t,1] = (f^{-1}(R_t))^{-1}(t,1] \ \varepsilon \tau, t \varepsilon \ [0,1].$$

Consequently g is a continuous function. Let h: $[0,1] \to [0,1]$ be defined by $h(x) = \frac{10}{9} x$ if $0 \leqslant x \leqslant 0.9$ and $h(x) = 1$ if $0.9 < x \leqslant 1$. Then hog: $(X,\tau) \to [0,1]$ is a continuous function and $(hog)(x_0) = h(g(x_0)) = 1$, $(hog)\ (X-U) = h(0) = 0$. This completes the proof of the theorem.

3.21 *Definition*

A fuzzy topological space (X,T) is called functionally Hausdorff iff for any two distinct fuzzy points p and q in X, there exists a continuous function f: $(X,T) \to [0,1]\ (L)$ such that for every $x \varepsilon X$

$$p(x) \leqslant f(x)(1-) \leqslant f(x)(0+) \leqslant q'(x).$$

3.22 *Theorem*

A topological space (X,τ) is functionally Hausdorff if and only if $(X,\omega(\tau))$ is functionally Hausdorff.

Proof: Let (X, τ) be a functionally Hausdorff space. Let p,q be two distinct fuzzy points in X. Then x_p, x_q are two distinct elements in X. Thus there exists a continuous function f: $(X,\tau) \to [0,1]$ such that $f(x_p) = 1$ and $f(x_q) = 0$. Define g:$(X, \omega(\tau)) \to [0,1]\ (L)$ as follows:

$$g(x)(t) = \chi_{f^{-1}(t,1]}(x),\ x \varepsilon X,\ t \varepsilon R.$$

Then it is easy to check that $g(x) \varepsilon [0,1]\ (L)$ for all $x \varepsilon X$, *i.e.*, g is a well defined function. Doing some calculations, one can verify the following inequalities: $p(x) \leqslant g(x)(1-) \leqslant g(x)(0+) \leqslant q'(x)$ for all $x \varepsilon X$. Since $g^{-1}(R_t) = \chi_{f^{-1}(t,1]} \varepsilon \omega(\tau)$ and $g^{-1}(L_t) = \chi_{f^{-1}[0,t]} \varepsilon \omega(\tau)$, $t \varepsilon [0,1]$, g is a continuous function.

Conversely, let $(X, \omega(\tau))$ be a functionally Hausdorff space. Let x_0, y_0 be two distinct elements in X. Take p,q to be the two fuzzy points in X for which $p(x_0) = q(y_0) = 0.9$. Then there exists a continuous function f:$(X, \omega(\tau)) \to [0,1](L)$ such that for any $x \varepsilon X$ we have $p(x) \leqslant f(x)(1-) \leqslant f(x)(0+) \leqslant q'(x)$. It follows that

$f(x_0)(1-) \geq 0.9$ and $f(y_0)(0+) \leq 0.1$. Define g: $(X,\tau) \to [0,1]$ by $g(x) = \inf \{t \varepsilon R: f(x)(t-) < t\}$. Then, as in the proof of Theorem 3.20, g is a well defined continuous function satisfying the inequalities: $g(x_0) > 0.9$ and $g(y_0) \leq 0.1$. Define h: $[0,1] \to [0,1]$ by $h(x) = 0$ if $0 \leq x \leq 0.1$, $h(x) = \frac{10}{8}(x-0.1)$ if $0.1 < x < 0.9$, and $h(x) = 1$ if $0.9 \leq x \leq 1$. Then hog: $(X,\tau) \to [0,1]$ is a continuous function satisfying: (hog) $(x_0) = h(g(x_0)) = 1$ and (hog) $(y_0) = h(g(y_0)) = 0$. This completes the proof of our theorem.

References

Chang, C.L. (1968) Fuzzy topological spaces, *J. Math. Anal. Appl.* **24:** 182-190.

Conrad, F. (1980) Fuzzy topological concepts, *J. Math. Anal. Appl.* **74:** 432-440.

Hutton, B. (1975) Normality in fuzzy topological spaces, *J. Math. Anal. Appl.* **50:** 74-79.

Hutton, B. (1977) Uniformities on fuzzy topological spaces, *J. Math. Anal. Appl.* **58:** 559-571.

Hutton, B. and **Reilly, I.** (1980) Separation axiom in fuzzy topological spaces, Fuzzy Sets and Systems, **3:** 93-104.

Ming, Pu-Pao and **Ming, Liu Ying** (1980) Fuzzy topology, *I.J. Math. Anal. Appl.* **76:** 571-599.

Sarkar, M. (1981) On fuzzy topological spaces, *J. Math. Anal. Appl.* **79:** 384-394.

Srivastava, P., Lal, A.N. and **Srivastava, A.K.** (1981) Fuzzy Hausdorff topological spaces, *J. Math. Anal. Appl.* **81:** 497-506.

Tong, H. (1952) Some characterizations of normal and perfectly normal spaces, *Duke Math. J.* **19:** p. 289-292.

Weiss, M.D. (1975) Fixed points, separation, and induced topologies for fuzzy sets, *J. Math. Anal. Appl.* **50:** 142-150.

Wong, C.K. (1974) Fuzzy points and local properties of fuzzy topology, *J. Math. Anal. Appl.* **46:** 316-328.

Zadeh, L.A. (1965) Fuzzy sets, *Inform. and Control.* **8:** 338-353.

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مسلمات الفصل السائبة والاستمرار السائب

علي أحمد فوره

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لنفرض أن (س، ت) فضاء توبولوجيا، ولنفرض أن د (ت) تدل على مجموعة جميع الدوال شبه المستمرة السفلية المعرفة من الفضاء التبولوجي س إلى الفترة المغلقة [صفر، ١]. لقد عرفنا في هذا البحث اثنتين من مسلمات الفصل السائبة وهي: الهاوسدورف الدالي والانتظام التام. وبعدها أثبتنا أن (أ) الفضاء التبولوجي (س، ت) يكون هاوسدورفي دالي إذا وفقط إذا كان الفضاء السائب (س، د (ت)) هاوسدورفي دالي و (ب) الفضاء التبولوجي (س، ت) يكون منتظما تماماً إذا وفقط إذا كان الفضاء السائب (س، د (ت)) منتظماً تماماً.