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Further Results on Stieltjes and Van Vleck Polynomials*

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ABSTRACT. Stieltjes and Van Vleck polynomials are the polynomial solutions of the generalized Lame's differential equation. The problem of determining the relative location of the zeros of such polynomials and the complex constants occurring in the said differential equation has recently been studied by Al-Rashed and Zaheer (1985) under quite general conditions by introducing the concept of reflector regions. The present study solves the corresponding problem for yet another form of the generalized Lame's differential equation and offers much more general versions of some results due to Zaheer and Alam. Furthermore, when applied to the standard form of the generalized Lame's differential equation, our results in this paper deduce the coresponding results of Al-Rashed and Zaheer (1985), including some due to Mardan, Bôcher, Klein, Pólya, Stieljes and Van Vleck as special cases.

1. Preliminaries

We begin with a brief introduction in which we present notation and terminology and some results from Al-Rashed and Zaheer (1985) for later use.

Given a subset S of the complex plane, we let $\varkappa(S)$, ∂S, $\mathring{S}$ denote the convex hull, the boundary and the interior of S, respectively. Throughout the present text, unless mentioned otherwise, K denotes a nonempty convex compact subset of the complex plane. Every supporting line of K passes through either a single supporting point of K or a whole line-segment of supporting points of K (For more details, refer to Levin 1964, p. 74-75). That is, each supporting line of K has at least

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one and at most two extreme points of K on it. It is well known (see Rudin 1973, Theorem 3.22, p. 71) that K has extreme points and that K is the convex hull of its extreme points. If K is not a singleton and if $z \notin K$, then we can always find two (possibly coincident) supporting lines of K, through the point z, each of which passes through a unique extreme point of K (the one nearest to z on each line) such that K lies in one of the angles between these lines. The angle between these lines containing K is called the *angle subtended* by K at z and is denoted by $\alpha(z) \equiv \alpha(z, K)$. Let v_z, $v_{z'}$, denote the two (possibly coincident) extreme points of K uniquely determined by z. If $v_z \neq v_{z'}$, the pair $(v_z, v_{z'})$ (resp. $(v_{z'}, v_z)$ is called the *crucial pair* of z with respect to K if traversing $\partial\triangle$, the boundary of the triangle with vertices at v_z, z and $v_{z'}$, starting from v_z (resp. $v_{z'}$) to z yields a clockwise orientation to $\partial\triangle$. In case $v_z = v_z'$, the crucial pair of z is defined by (v_z, v_z).

Thus we record: *To every point $z \notin K$, we can associate a unique crucial pair of extreme points of* K.

Given K and an angle ϕ $(0 < \phi < \pi)$, we define (see Marden 1966, p. 31) the *star-shaped* region $S(K,\phi)$by

$$S(K,\phi) = \{z \varepsilon \mathbf{C} \; | \alpha(z,K) \geq \varphi\},$$

where **C** denotes the set of all complex numbers. Clearly, $S(K,\phi)$ is bounded. If we now define

$$\alpha(z,k) = \begin{cases} \pi & \forall \quad z \varepsilon \partial K, \\ 2\pi & \forall \quad z \varepsilon \mathring{K}, \end{cases}$$

then $K \subset S(K,\phi)$ and $K = S(K,\pi)$. Obviously, $K = S(K,\phi)$ if K is a singleton. Observe that the line segment jointing u,v lies in $S(K, \phi)$ for all $u \varepsilon K$ and $v \varepsilon S(K, \phi)$. Also, it is easy to verify that $S(K, \phi) \subset S(K', \phi)$ if $K \subset K'$ and that $S(K, \phi) \subset S(K, \phi')$ if $\phi' \leq \phi$.

Next we recall the following definitions and properties (see Al-Rashed and Zaheer 1985):

Definition 1.1

Given K and a regular Jordan arc C not cutting K in the complex plane, we say that C is a *reflector arc* for K if the normal at every point $c \varepsilon C$ is along the bisector of the angle $\alpha(c,K)$. A closed reflector arc for K is called a *reflector curve* for K.

Definition 1.2

Given K, we say that K is of *reflecting type* if it has a unique convex reflector curve (C_z, say), enclosing K, through every point $z \notin K$. We denote by $\mathcal{F}$ the family of all nonempty convex compact subsets of reflecting type.

Remark 1.3

(i) It may be noted, as an obvious consequence of Definition (1.2), that, for $K \varepsilon \mathcal{F}$, any two reflector curves C_z, $C_{z'}$ (say) for K are either identical or disjoint.

(ii) Every closed line segment joining a,b, belongs to $\varkappa$ (C_z being the ellipse through z with focii at a and b). Every closed disk with center at c is also in $\mathcal{F}$ (In this case, C_z is the concentric circle through z.).

Next we recall a lemma (see Al-Rashed and Zaheer 1985, lemma 2.2) which assures the existence of reflector curves for certain types of convex sets in addition to closed line segments and closed disks.

Lemma 1.4

If $K_n = \varkappa(P_n)$, *where* P_n *is a closed polygon with* n *vertices, then*

$$K_n \ \varepsilon \ \mathcal{F} \ \text{for all } n \geq 3.$$

2. Main Results

A generalized Lame's differential equation is a second order linear differential equation of the form

$$(2.1) \qquad \frac{d^2w}{dz^2} + \left[\sum_{j=1}^{p} \alpha_j/(z - a_j)\right] \frac{dw}{dz} + \frac{\phi(z)}{\prod_{j=1}^{p} (z - \alpha_j)} w = 0,$$

where $\phi(z)$ is a polynomial of degree at most (p-2) and where α_j, a_j are complex constants. It is known (see Heine 1878 or Marden 1966, P.36) that there exist at most (C(n+p−2, p−2) polynomial solutions V(z) (called *Van Vleck polynomials*) such that for $\phi(z) = V(z)$, the equation (2.1) has a polynomial solution S(z) of degree n (called *Stieltjes polynomials*).

Let us consider the differential equation (see Zaheer and Alam 1977).

$$(2.2) \qquad \frac{d^2w}{dz^2} + \left[\sum_{j=1}^{p} \alpha_j \left\{\frac{\prod_{t=1}^{n_j-1} (z-b_{jt})}{\prod_{s=1}^{n_j} (z-a_{js})}\right\}\right] \frac{dw}{dz} + \frac{\phi(z)}{\prod_{j=1}^{p} \prod_{s=1}^{n_j} (z-a_{js})} w = 0,$$

where $\phi(z)$ is a polynomial of degree at most $(n_1 + n_2 + \cdots + n_p - 2)$ and where α_j, a_{js} and b_{jt} are complex constants. It may be noted that the differential equation (2.2.) can always be written in the form (2.1) by expressing each fraction (in the coefficient of dw/dz) into its partial fractions and that (2.2) is indeed of the form (2.1) if $n_j = 1$ for all j. It may also be observed (as in case of (2.1)) that there exist at most

$$C(n+n_1+\cdots+n_p-2,\ n_1+n_2+\cdots+n_p-2)$$

polynomials V(z) such that for $\phi(z) = V(z)$, the differential equation (2.2) has a polynomial solution S(z) of degree n. Consequently, there do exist Stieltjes polynomials S(z) and Van Vleck polynomials V(z) associated with the differential equation (2.2).

For convenience, we adopt the following notations:

$$(2.3) \qquad f_j(z) = \Pi_{t=1}^{n_j-1} (z-b_{jt}); g_j(z) = \Pi_{s=1}^{n_j-1} (z-a_{js}); h_j(z) = \frac{f_j(z)}{g_j(z)}$$

for all $j=1,2,\ldots,p$ (with $f_j(z) \equiv 1$ for $n_j = 1$),

$$(2.4) \qquad F(z) = \sum_{j=1}^{p} \alpha_j h_j(z).$$

Also, we write $q = \max\{n_1, n_2, \ldots, n_p\}$.

Next, we recall the following results, concerning the zeros of S(z) and V(z), to be used in the proof of our main theorems.

Lemma 2.1

(Zaheer and Alam 1977, Lemma 2.1). *Every zero* z_k *of an nth degree stieltjes polynomial* S(z), *in relation to* (2.2), *is either one of the points* a_{js} $(1 \le j \le p, 1 \le s \le n_j)$ *or it satisfies the equation*

$$(2.5) \qquad \tfrac{1}{2}F(z_k) + \sum_{j \neq k, j=1}^{n} \frac{1}{z_k - z_j} = 0.$$

Lemma 2.2

(Zaheer and Alam 1977, Lemma 2.2). *Every zero* t_k *of a Van Vleck polynomial* V(z), *associated with the differential equation* (2.2), *if not an* a_{js}, *is either one of the zeros of* S′(z) *or it satisfies the equation*

$$(2.6) \qquad F(t_k) + \sum_{j=1}^{n-1} \frac{1}{t_k - z'_j} = 0,$$

where the z'_j *are the zeros of* $S'(z)$.

Now, given $K\varepsilon \mathcal{F}$, $\gamma\varepsilon[0,\pi/2)$ and an integer $q\geq 1$, we write $S_{\gamma,q} \equiv S\,(K,(\pi-2\gamma)/(2q-1))$ and let C_z denote the unique reflector curve through $z\notin K$, with $R_z=\varkappa(C_z)$ termed as the *reflector region* for K determined by z. If $K_{\gamma,q}$ denotes the intersection of all the reflector regions R_z containing $S_{\gamma,q}$, we see that $K_{\gamma,q}$ is a nonempty convex compact subset such that

$$(2.7) \qquad K \subset S_{\gamma,q} \subset K_{\gamma,q}\,.$$

Note that $K \subsetneq S_{\gamma,q}$ for $\gamma>0$ and that $K=S_{0,1}$. However, for $\gamma=0$ and $q=1$, we know (Al-Rashed and Zaheer 1985) that

$$(2.8) \qquad K = S_{0,1} = K_{0,1}\,.$$

Now, we are ready to prove our main theorems concerning the zeros of S(z) and V(z), associated with (2.2). The results that we obtain are valid for both the forms (2.1) and (2.2), whereas the corresponding known results (see Al-Rashed and Zaheer 1985) apply only to the form (2.1) and turn out as corollaries of our results here.

Theorem 2.3

In the differential equation (2.2), *if*

$$|\arg\alpha_j| \leq \gamma < \pi/2 \ \forall \ j=1,2,\ldots,p,$$

and if all the points a_{js} *and* b_{jt} *(occurring in* (2.2)*) lie in* K ($K\varepsilon \mathcal{F}$), *then the zeros of each stieltjes polynomial* S(z) *lie in* $K_{\gamma,q}$.

Proof

Let S(z) be an nth degrees stieltjes polynomial, associated with (2.2), with zeros $z_1,z_2,\ldots,z_n$. On the contrary, suppose that $z_1,\ldots,z_m$ $(1\leq m\leq n)$, after a reindexing if necessary, do not belong to $K_{\gamma,q}$. Consider the C_{z_i} for $i=1,2,..,m$ (possible, since $K\varepsilon \mathcal{F}$) and let C_{z_1} (after reindexing again, if necessary) be the outermost one (*cf.* Remark (1.3)(i)). Since $z_1\notin K_{\gamma,q}$, we note that $z_1\notin R_z$ for some $R_z \supset S_{\gamma,q}$. In other words, $K_{\gamma,q} \subset R_z \subsetneq R_{z_1}$ and we have that (cf. (2.7)).

$$K \subset S_{\gamma,q} \subset K_{\gamma,q} \subsetneq R_{z_1}.$$

Let H be the open half plane (containing K) with ∂H as the tangent (L,say) at z_1 to C_{z_1}. Obviously, $z_1 \notin H$ and $z_j \notin H$ for $j \neq 1$. Let $2\psi = \alpha(z_1, K)$. Then

(2.9) $$0 < 2\psi < (\pi - 2\gamma)/(2q-1).$$

Now, consider a point u_1 ($\neq z_1$) on the bisector of the angle $\alpha(z_1, K)$ and let $\phi = \arg(u_1 - z_1)$. Since z_1 is a zero of S(z), it satisifes (2.5) with k=1. That is,

(2.10) $$-\tfrac{1}{2}F(z_1) + \sum_{j=2}^{n} \frac{1}{\bar{z}_j - \bar{z}_1} = 0.$$

Let us observe that, for all $1 \leq j \leq p$, $1 \leq t \leq n_j - 1$ and

$1 \leq s \leq n_j$, we have

$$-2\psi \leq \arg\left(\frac{b_{jt} - z_1}{a_{js} - z_1} \right) \leq 2\psi.$$

This implies that

(2.11) $$-2\psi \leq \arg\left(\frac{\bar{b}_{jt} - \bar{z}_1}{\bar{a}_{js} - \bar{z}_1} \right) \leq 2\psi.$$

Further, we have

(2.12) $$\phi - \psi \leq \arg\left(\frac{1}{\bar{a}_{js} - \bar{z}_1} \right) \leq \phi + \psi.$$

From (2.11), (2.12) and (2.3), we see that

$$\phi - (2n_j - 1)\psi \leq \arg \overline{-h_j(z_1)} \leq \phi + (2n_j - 1)\psi$$

and hence (*cf.* (2.9) and definition of q) that

$$\phi - \frac{\pi - 2\gamma}{1} < \arg(-h_j(z_1)) < \phi + \frac{\pi - 2\gamma}{2}$$

But the inequalities

$$-\gamma \leq \arg \bar{\alpha}_j \leq \gamma \quad (j = 1, 2, \ldots p)$$

imply that

(2.13) $$\phi - \pi/2 < \arg\left(-\tfrac{1}{2}\bar{\alpha}_j \overline{h_j(z_1)}\right) < \phi + \pi/2 \quad \forall j = 1, 2, \ldots, p.$$

Next, we observe (since $z_1 \notin H$ and $z_j \notin H$ for j=2,3,...n) that

$$(2.14) \qquad \phi - \pi/2 < \arg\left(\frac{1}{\bar{z}_j - \bar{z}_1}\right) < \phi + \pi/2 \qquad \forall\, j = 2,3,\ldots,n.$$

Now (2.13) and (2.14) furnish the fact that every term in (2.10) satisfies the inequality

$$(2.15) \qquad \phi - \pi/2 < \arg z < \phi + \pi/2.$$

Therefore,

$$\sum_{j=1}^{p} \{ -\tfrac{1}{2}\bar{\alpha}_j \overline{h_j(z_1)} \} + \sum_{j=2}^{n} \frac{1}{\bar{z}_j - \bar{z}_1} \neq 0,$$

which contradicts (2.10) (*cf.* (2.4)). The proof is now complete.

Theorem 2.4

Under the notations and hypotheses of Theorem (2.3), *the zeros of each Van Vleck Polynomial lie in* $K_{\gamma,q}$.

Proof

Let V(z) denote a Van Vleck polynomial corresponding to an nth degree stieltjes polynomial S(z) and let z'_j $(1 \leq j \leq n-1)$ be the zeros of S′(z), the derivative of S(z). If t_k is a zero of V(z), lemma 2.2 says that either $t_k = z'_j$ for some j (in which case $t_k \in K_{\gamma,q}$ by Theorem (2.3) and Lucas theorem (see Marden 1966, Theorem (6.1) or Lucas 1874), since $K_{\gamma,q}$ is convex), or, else, t_k satisfies (2.6). In the latter case, if (on the contrary) some zeros of V(z) lie outside $K_{\gamma,q}$ we proceed as in the proof of Theorem (2.3) and obtain a zero t_1 (say) of V(z) outside $K_{\gamma,q}$ such that $K \subset S_{\gamma,q} \subset K_{\gamma,q} \subsetneqq R_{t_1}$ and

$$(2.16) \qquad -\overline{F(t)} + \sum_{j=1}^{n-1} \frac{1}{\bar{z}'_j - \bar{t}_1} = 0.$$

Continuing, further, with the techniques and steps of the proof of Theorem (2.3), with z_1 replaced by t_1 and z_j $(2 \leq j \leq n)$ replaced by z'_j $(1 \leq j \leq n-1)$, we obtain a contradiction to (2.16) and this completes the proof.

The following results are corollaries of Theoremes (2.3) and (2.4).

Corollay 2.5

(Al-Rashed and Zaheer 1985, Theroem 3.1). *In the differential equation* (2.1), *if*

$$|\arg \alpha_j| \leq \gamma < \pi/2 \qquad \forall \quad j=1,2,\ldots,p,$$

and if the points a_j *lie in* K(Kε $\mathcal{F}$), *then the zeros of each stieljes (resp. Van Vleck) polynomial lie in* K_γ, *where* K_γ *is the smallest reflector region of* K *containing the set* $S_\gamma \equiv S$ (K, $\pi-2\gamma$).

Proof

Let $n_j=1$ for all j (so that q=1), then (2.2) reduces to (2.1), with the a_{j1} (in (2.2)) corresponding to the constants a_j in (2.1). Consequently, $S_{\gamma,q}$ of Theorems (2.3) and (2.4) is indeed $S_\gamma \equiv S$ (K, $\pi-2\gamma$) and the proof of corollary (2.5) is obvious.

Corollary 2.6

(Zaheer 1976, Theorems 2.1 and 2.2). *Under the notations and hypotheses of Theorem* (2.3), *if* K *is the line seqment* σ *joining* c_1 *and* c_2, *then the zeros of each Stieltjes (resp. Van Vleck) polynomial lie in the region*

$$\sigma^* = \{z\varepsilon \mathbf{C}|\ |z-c_1| + |z-c_2| \leq |c_1-c_2| \sec \frac{(q-1)\pi+\gamma}{2q-1}\}$$

Proof

If we put $\alpha=(\pi-2\gamma)/(2q-1)$, then $0<\alpha\leq \pi/(2q-1) \leq \pi$. By Remark (1.3)(ii), K=σ ε $\mathcal{F}$ and

$$S_{\gamma,q} = \begin{cases} D_1 \cup D_2 & \text{if } 0< \alpha < \pi/2, \\ D_1 \cap D_2 & \text{if } \pi/2 \leq \alpha \leq \pi, \end{cases}$$

where ∂D_1 and ∂D_2 are circle, both passing through c_1 and c_2 with radius equal to $\frac{1}{2}|c_1-c_2|\csc\alpha$. Let c be the midpoint of σ and z_0, z'_0 the two points at which the perpendicular bisector of σ meets $S_{\gamma,q}$. Then, for $0<\alpha \leq \pi$,

$$|c-z_0| = |c-z'_0| = \tfrac{1}{2}|c_1-c_2| \tan(\pi/2 - \alpha/2) = b \text{ (say)}.$$

If $a = \frac{1}{2}|c_1-c_2|\sec(\pi/2 - \alpha/2)$, then the ellipse, E_0, with focii at c_1, c_2 and passing through z_0, z'_0, has its semimajor and semiminor axes as a and b, respectively. Consequently,

$$E_0 = \{ z\varepsilon\mathbf{C} \mid |z-c_1| + |z-c_2| = 2a \}$$

is the smallest of all the confocal ellipses (with focii c_1, c_2) containing $S_{\gamma,q}$. Since C_z, for every $z\notin K$, is the ellipse through z with focii at c_1, c_2 ((*cf.* Remark (1.3(ii)), we see that $\partial K_{\gamma,q}$ is the ellipse E_0 if one observes that

$$(\pi - \alpha)/2 = \{(q-1)\pi+\gamma\} / (2q-1).$$

Now, Theorems (2.3) and (2.4) complete the proof.

Corollary 2.7

(Zaheer and Alam 1977, Corollary (2.5)). *Under the notations and hypotheses of Theorem* (2.3), *if $K=D(c,r)$, the disk with center* c *and radius* r, *then zeros of each Siteltjes (resp. Van Vleck) polynomial lie in* D(c,r'), *where*

$$r'=r \sec [\{(q-1)\pi+\gamma\}/(2q-1)].$$

Proof

We know that K $=D(c,r)\varepsilon\mathcal{F}$, that

$$S_{\gamma,q} = D(c,r \sec(\pi/2-\alpha/2)),$$

where $\alpha = (\pi-2\gamma)/(2q-1)$, and that

$$R_z = D(c,|z-c|) \ \forall \ z\varepsilon D(c,r).$$

Hence $K_{\gamma,q}$ is the intersection of all the disks $D(c,\rho)$ with $\rho\geqslant r \sec(\pi/2-\alpha/2)$ (resp. $\rho>r$) if $\alpha\leqslant \pi$ (resp. $\alpha=\pi$).

Therefore, in both cases

$$K_{\gamma,q}=D(c,r \sec (\pi/2-\alpha/2)),$$

where $\pi/2-\alpha/2= \{(q-1)\pi+\gamma\}/(2q-1)$. Theorems (2.3) and (2.4) now establish the corollary.

For q=1, corollary (2.6) (resp. corollary (2.7)) is essentially a result due to Marden (1931, Theorem 6(b)) (resp. Marden (1931, Theorem 4) or Marden (1966, Theorem (9,1)). It is already shown (see Al-Rashed and Zaheer 1985, Corollaries 3.2–3.4) that corollary (2.5) includes in it some results due to Marden (1931, Theorems 4 and 6(b)), Marden (1966, Exercise 4, p.41), Bôcher (1894) (or Marden 1966, Exercise 3, p.41), Klein (1894) (or Marden 1966, Exercise 3, p.41]), Pólya (1912) (or Marden 1966, Exercise 3,p.41), Stieltjes (1885) and Van Vleck (1898) (or Marden 1966, Exercises 1 and 2, p.41).

Finally, we state the following result concerning solutions of the system of equations (2.5) for $k=1,2,\dots,n$.

Theorem 2.8

In the system of equation (2.5) *for* $k=1,2,\dots,n$ *in the variables* $z_1,z_2,\dots,z_n$, *let the constants* a_{js}, b_{jt} *and* α_j *satisfy the conditions of Theorem* (2.3). *Then every solution* z_k, $k=1,2,\dots,n$, *of this system lies in* $K_{\gamma,q}$, *where* $q=\max\{n_1,n_2,\dots,n_p\}$.

Proof

The proof is exactly the same as that of Theorem (2.3) except that $z_1,z_2,\dots,z_n$ now designate a solution of the system of equations (2.5) for $k=1,2,\dots,n$.

As seen in the proof of corollary (2.5), if $n_j=1$ for all j(so that $q=1$) then (2.2) reduces to (2.1) with a_{j1} corresponding to a_j. Therefore, $S_{\gamma,q}=S(K,\pi-2\gamma)$, $K_{\gamma,q}=K_\gamma$ (see Corollary (2.5)) and (2.5) reduces to the equation

$$(2.17)\qquad \sum_{j=1}^{p} \frac{\alpha_j/2}{z_k-a_j} + \sum_{j\neq k,\, j=1}^{n} \frac{1}{(z_k-z_j)} = 0$$

In view of this, the following result is immediate from Theorem (2.8) when $q=1$.

Corollary 2.9

(Al-Rashed and Zaheer 1985, Theorem 3.5). *In the system of equations* (2.17) *for* $k=1,2,\dots,n$ *in the variables* $z_1,z_2,\dots z_n$, *let the constants* α_j *and* a_j *satisfy the hypotheses of corollary* (2.5). *Then every solution* z_k, $k=1,2,\dots,n$, *of this sytem lies in* k_γ, *where* K_γ *is as defined in corollary* (2.5).

Remarks 2.10

(i) Theorem (2.8) and, hence, Corollary (2.9) may also be regarded as a generalization of Lucas' theorem (see Lucas (1874) or Marden 1966, Theorem (6.1)') to systems of partial fraction sums (Take $q=1$, $\gamma=0$ and $n=1$).

(ii) Theorems (2.3) and (2.8) (as well as Corollaries (2.5) and 2.9)) have physical applications in the theories of classical orthogonal polynomials (see Szegö (1967, pp. 29,44,58-63,74,155,), Marden (1966, p.41), Alam (1979, p.203)), equilibrium of gravitational (electrostatic or electromagnetic) forces (see Marden (1966, pp. 8,33,37), (Kellog 1953, p. 10)) and stagnation points in a two-dimensional fluid motion (see Milne-Thomson 1968, pp.209,210,357). We do not intend to discuss these here. However, a detailed discussion about such applications can be found in Al-Rashed and Zaheer (1985), Section 4.

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قسم الرياضيات ـ جامعة الملك سعود ـ الرياض ـ
المملكة العربية السعودية

كثيرات حدود ستلجى وفان فليك هي حلول لمعادلة لامي التفاضلية المعممة. أن مسألة تعيين الموقع المحلي لأصفار كثيرات الحدود تلك والثوابت المركبة في هذه المعادلة التفاضلية قد درسها حديثاً كل من الراشد وظهير (١٩٨٥) تحت شروط عامة وذلك عن طريق تقديم مفهوم المناطق العاكسة. هذا البحث يحل المسألة المماثلة لصيغة أخرى لمعادلة لامى التفاضلية المعممة ويعطى صيغ أكثر تعمياً لبعض نتائج كل من ظهير وعلام. بالإضافة إلى ذلك، هذا الحل عندما يطبق على الصيغة التقليدية لمعادلة لامى التفاضلية المعممة، فإن النتائج التي حصلنا عليها في هذا البحث تعطى النتائج المماثلة للراشد وظهير (١٩٨٥)، بما في ذلك بعض النتائج لكل من ماردن وبوشر وكلاين وبوليا وستلجى وفان فليك كحالات خاصة.