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The Smoothness of the Heat Equation Solution in a Singular Domain of R^{n+1}

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ABSTRACT. It is well known that the heat operator is an isomorphism from a certain Sobolev space onto the space of square integrable functions, as long as the domain of these functions is smooth (cylindrical). We prove that the same result remains true when the domain is any convex polyhedron. The proof uses the elliptic regularization method and some results of Sadallah (1976).

1. Introduction

Let Ω be any bounded convex polyhedral domain in R^{n+1}, generated by the variables (t,x), where $x = (x_1, \ldots, x_n)$. We denote by Γ the boundary of Ω and by Γ_N the part of Γ which satisfies:

$$(T,x) \in \Gamma_N \ , \text{ where } T = \sup_{(t,x) \in \Gamma} t$$

For the definition and propreties of convex polyhedral domains, see for example Glazman and Liubitch (1974).

We are concerned with the following problem:

The existence and the uniqueness of the solution $u \in H^{1,2}(\Omega)$ of:

$$(P) \begin{cases} d_t u - D^2_{x_1} u - \cdots - D^2_{x_n} u = f \in L^2(\Omega) \\ u|_{\Gamma - \Gamma_N} = 0 \end{cases}$$

where $H^{1,2}(\Omega) = \{u \in H^1(\Omega), D_{xi} D_{xj} u \in L^2(\Omega); i,j = 1,..,n\}$.

$L^2(\Omega)$ being the space of the functions the squares of which are integrable in Ω.

Also, the space (of Sobolev) $H^{1,2}(\Omega)$ is a space which, when equipped with the norm:

$$\|u\|_{H^{1,2}(\Omega)} = (\|u\|^2_{H^1(\Omega)} + \sum_{i,j=1}^{n} \|D_{x_j} D_{x_i} u\|^2)^{1/2}$$

becomes a Hilbert space.

We note that $\|\cdot\|$ (without index) denotes throughout this paper the L^2 norm.

We remark that the boundary condition of problem (P) is justified by the theory of symmetric systems of Lax and Phillips (1960) and the theory of Kohn and Niremberg (1967).

Problem (P) has been solved in the case of one space variable ($n = 1$) by the elliptic regularization method (Sadallah 1976) and by the approximation method of the domain Ω, (Sadallah 1983). In present paper the former approach is used because it agrees better with the method of the mathematical induction (on n).

Our main result is similar to the one obtained with one space variable, this is:

Theorem 1.1. When Ω is a bounded convex polyhedral domain of R^{n+1}, the heat operator $D_t - \Delta$ is an isomorphism from

$$H^{1,2}_c(\Omega) = \{u \in H^{1,2}(\Omega)\ ,\ u|_{\Gamma-\Gamma_N} = 0\} \text{ onto } L^2(\Omega)$$

Observe that this result is not true when Ω is not convex, even if Ω is cylindrical with respect to the time variable. Indeed, the non convexity of Ω allows some singular solutions to appear, hence, making the space of the solutions larger than $H^{1,2}_c(\Omega)$ [see, for instance, Moussaoui and Sadallah (1981, 1984)]. The proof of theorem 1.1 will be achieved by induction (on n). Since the case $n=1$ has been studied, See Sadallah (1976), it remains to show how to pass from the dimension n to the dimension $n + 1$.

For simplicity, we restrict ourselves to exhibit the details of the passing from one space dimension to two dimensions, knowing that the general case may be treated similarly.

2. The study of the regularized problem

From now on, (x, y) will denote the two space variables. We introduce the following hypothesis on the boundary of Ω :

(H) $\{$ Γ_N is reduced either to one point or to one segment (of R)

This condition permits us to replace the boundary condition $u|_{\Gamma-\Gamma_N} = 0$, valid almost everywhere, by $u|_{\Gamma} = 0$. In general case, (H) means that dim $\Gamma_N < n$.

Now, consider the following (regularized) problem which is to solve under the condition (H) :

$$(P_\varepsilon) \quad \begin{cases} \text{Existence and uniqueness of the solution} \\ u_\varepsilon \in H^2(\Omega) \cap H^1_o(\Omega) \text{ of the equation:} \\ D_t u_\varepsilon - \varepsilon D_t^2 u_\varepsilon - D_x^2 u_\varepsilon - D_y^2 u_\varepsilon = f \in L^2(\Omega), \\ \text{where } \varepsilon > 0. \end{cases}$$

2.1. Resolution of problem (P_ε)

Theorem 2.1. For each positive ε and $f \in L^2(\Omega)$, problem (P_ε) has one and only one solution.

Proof: The domain Ω has the continuation proprety of Nečas (1967) because Ω is convex and bounded. Hence, the operator $D_t : H^2(\Omega) \to L^2(\Omega)$ is compact, consequently, the operators $\varepsilon D_t^2 + D_x^2 + D_y^2$ and $D_t - \varepsilon D_t^2 - D_x^2 - D_y^2$ have the same index.

Observe that the Sobolev spaces, the convexity of domains and the condition (H) are preserved under the change of variables:

$$\begin{array}{ccc} \Omega & \longrightarrow & \Omega_\varepsilon \\ (t,x,y) & \mapsto & (t/\sqrt{\varepsilon}, x, y) \end{array}$$

Moreover the operator $\varepsilon D_t^2 + D_x^2 + D_y^2$ becomes $\triangle = D_t^2 + D_x^2 + D_y^2$. Therefore, Pbm ($P_\varepsilon$) is equivalent to the following one :

$$(P'_\varepsilon) \quad \begin{cases} \text{Existence and uniqueness of the solution} \\ v_\varepsilon \in H^2(\Omega_\varepsilon) \cap H^1_o(\Omega_\varepsilon) \text{ of the equation:} \\ D_t^2 v_\varepsilon + D_x^2 v_\varepsilon + D_y^2 v_\varepsilon = g \in L^2(\Omega_\varepsilon) \end{cases}$$

Indeed, it suffices to put:

$$u_\varepsilon(t,x,y) = v_\varepsilon(t/\sqrt{\varepsilon}, x, y)$$
$$f(t,x,y) = g(t/\sqrt{\varepsilon}, x, y)$$

Finally, we know—Kadlec (1964)—that problem (P'_ε) has a unique solution.

C.Q.D.

2.2. A priori estimate

We shall look for an a priori estimate of the type of the second fundamental estimate of Ladyzhenskaya and Ural'tseva (1968). More specifically, we shall prove the following

Proposition 2.2. There exists a positive constant C, which only depends on the geometry of Ω, such that:

$$\|u_\varepsilon\|_{H^{1,2}(\Omega)} + \varepsilon \, \|D_t^2 u_\varepsilon\| + \sqrt{\varepsilon}\,(\|D_t D_x u_\varepsilon\| + \|D_t D_y u_\varepsilon\|) \leq C \, \|f\|$$

where u_ε is the solution of problem (P_ε).

To derive this basic inequality, we need the results of the four following lemmas.

Let $(\cdot\,,\,\cdot)$ be the scalar product of $L^2(\Omega)$.

Lemma 2.3. One has:

$$(D_x^2\, u_\varepsilon,\ D_y^2\, u_\varepsilon) = \|D_x D_y u_\varepsilon\|^2$$
$$(D_t^2\, u_\varepsilon,\ D_x^2\, u_\varepsilon) = \|D_t D_x u_\varepsilon\|^2$$
$$(D_t^2\, u_\varepsilon,\ D_y^2\, u_\varepsilon) = \|D_t D_y u_\varepsilon\|^2$$

It is a result of Grisvard (1975).

Lemma 2.4. There exists a positive constant L which only depends on the geometry of Ω, such that:

$$\|u_\varepsilon\|^2 + \|D_x u_\varepsilon\|^2 + \|D_y u_\varepsilon\|^2 \leq L \, \|f\|^2$$

Indeed, the boundary condition $u_{\varepsilon|\Gamma} = 0$ implies:

$$(f, u_\varepsilon) = \varepsilon \, \|D_t u_\varepsilon\|^2 + \|D_x u_\varepsilon\|^2 + \|D_y u_\varepsilon\|^2$$

Therefore:

$$\|D_x u_\varepsilon\|^2 + \|D_y u_\varepsilon\|^2 \leq \ell \, \|u_\varepsilon\|^2 + \frac{1}{\ell} \cdot \|f\|^2, \ \forall \ \ell > 0.$$

We obtain the estimate of the lemma by applying Poincaré inequality (here, $u_\varepsilon \in H_0^1(\Omega)$ and Ω is bounded) and choosing ℓ small enough.

Lemma 2.5. When $n = 1$, there exists a positive constant M which only depends on the geometry of Ω, such that:

$$2\varepsilon|(D_t u_\varepsilon, D_t^2 u_\varepsilon)| + 2|(D_t u_\varepsilon, D_x^2 u_\varepsilon)| \leq \ell M \|D_x^2 u_\varepsilon\|^2 + \frac{M}{\ell} \cdot \|D_x u_\varepsilon\|^2$$
$$\leq \ell M \|D_x^2 u_\varepsilon\|^2 + \frac{M}{\ell} \cdot \|f\|^2, \ \forall\ \ell > 0$$

for all positive ε small enough, Here, x is the (unique) space variable.

This lemma is a result of Sadallah (1976).

The analogous of this lemma for $n = 2$ (which is required for the induction) is the following

Lemma 2.6. There exists a positive constant M', which only depends on the geometry of Ω, such that:

$$2\varepsilon|(D_t u_\varepsilon, D_t^2 u_\varepsilon)| + 2|(D_t u_\varepsilon, D_x^2 u_\varepsilon)| + 2|(D_t u_\varepsilon, D_y^2 u_\varepsilon)| \leq$$
$$\leq \ell M' (\|D_x^2 u_\varepsilon\|^2 + \|D_y^2 u_\varepsilon\|^2 + \frac{M'}{\ell} (\|D_x u_\varepsilon\|^2 + \|D_y u_\varepsilon\|^2)$$
$$\leq \ell M' (\|D_x^2 u_\varepsilon\|^2 + \|D_y^2 u_\varepsilon\|^2 + \frac{M'}{\ell} \cdot \|f\|^2, \ \forall\ \ell > 0$$

for all positive ε small enough.

Proof: Let us set $\Omega_y = \Omega \cap \{y = \text{constant}\}$, $\Omega_x = \Omega \cap \{x = \text{constant}\}$ and Γ_y (resp. Γ_x) the boundary of Ω_y (resp. Ω_x). Hence, Ω_y (resp. Ω_x) is a polygonal domain in (t,x) (resp. (t,y)) and we have, for almost every y (resp. x): $u|_{\Omega_y} \in H^2(\Omega_y) \cap H_o^1(\Omega_y)$ (resp. $u|_{\Omega_x} \in H^2(\Omega_x) \cap H_o^1(\Omega_x)$).

Then, the first inequality of Lemma 2.5, yields:

$$2\varepsilon|(D_t u_\varepsilon, D_t^2 u_\varepsilon)| + 2|(D_t u_\varepsilon, D_x^2 u_\varepsilon)| + 2|(D_t u_\varepsilon, D_y^2 u_\varepsilon)| =$$
$$| \int_y 2\varepsilon(D_t u_\varepsilon, D_t^2 u_\varepsilon)_{L^2(\Omega_y)} dy| + | \int_y 2(D_t u_\varepsilon, D_x^2 u_\varepsilon)_{L^2(\Omega_y)} dy| +$$
$$+ | \int_x 2(D_t u_\varepsilon, D_y^2 u_\varepsilon)_{L^2(\Omega_x)} dx|$$
$$\leq | \int_y \{ \frac{M(y)}{\ell} \|D_x u_\varepsilon\|^2_{L^2(\Omega_y)} + M(y) \cdot \ell \ \|D_x^2 u_\varepsilon\|^2_{L^2(\Omega_y)}\} \ dy +$$
$$+ | \int_x \{ \frac{M(x)}{\ell} \|D_y u_\varepsilon\|^2_{L^2(\Omega_x)} + M(x) \cdot \ell \ \|D_y^2 u_\varepsilon\|^2_{L^2(\Omega_x)}\} \ dx$$
$$\leq \frac{M'}{l} (\|D_x u_\varepsilon\|^2 + \|D_y u_\varepsilon\|^2) + M'.l\ (\|D_x^2 u_\varepsilon\|^2 + \|D_y^2 u_\varepsilon\|^2)$$
$$\leq \frac{M'}{l} \cdot \|f\|^2 + M'.l(\|D_x^2 u_\varepsilon\|^2 + \|D_y^2 u_\varepsilon\|^2), \forall\ \ell > 0 \qquad \text{(Lemma 2.4)},$$

where $M' = \sup (M(x) + M(y))$. This supremum is bounded because the functions $M(x)$ and $M(y)$ only depend on the positions of the faces (of Ω) with respect to the coordinates system and because the number of these faces is finite. This proves the lemma.

Proof of Propo. 2.2. Expanding the expression

$$\|f\|^2 = \|D_t u_\varepsilon - \varepsilon D_t^2 u_\varepsilon - D_x^2 u_\varepsilon - D_y^2 u_\varepsilon\|^2,$$

we get, by lemma 2.3.:

$$\begin{aligned} \|f\|^2 = {} & \|D_t u_\varepsilon\|^2 + \varepsilon^2 \|D_t^2 u_\varepsilon\|^2 + 2\varepsilon \|D_t D_x u_\varepsilon\|^2 + \\ & + 2\varepsilon \|D_t D_y u_\varepsilon\|^2 + \|D_x^2 u_\varepsilon\|^2 + \|D_y^2 u_\varepsilon\|^2 + \\ & + 2 \|D_x D_y u_\varepsilon\|^2 - 2 [\varepsilon (D_t u_\varepsilon, D_t^2 u_\varepsilon) + (D_t u_\varepsilon, D_x^2 u_\varepsilon) + \\ & + (D_t u_\varepsilon, D_y^2 u_\varepsilon)]. \end{aligned}$$

Furthermore, lemma 2.6 gives the estimate

$$\begin{aligned} & \|D_t u_\varepsilon\|^2 + \varepsilon^2 \|D_t^2 u_\varepsilon\|^2 + 2\varepsilon \|D_t D_x u_\varepsilon\|^2 + 2\varepsilon \|D_t D_y u_\varepsilon\|^2 + \\ & \quad + 2 \|D_x D_y u_\varepsilon\|^2 + (1 - lM') (\|D_x^2 u_\varepsilon\|^2 + \|D_y^2 u_\varepsilon\|^2) \\ & \quad \leqslant \left(\frac{M'}{l} + 1\right) \|f\|^2, \ \forall\, \ell > 0. \end{aligned}$$

The proof ends by choosing l small enough (e.g. $l = 1/2M'$) in the previous inequality and making use of lemma 2.4

C.Q.D.

3. Proof of Theorem 1.1.

We shall consider the two possible cases depending on whether hypothesis (H) is verified or not.

3.1. Case one: Ω verifies the condition (H).

For all $\varepsilon > 0$ and $f \in L^2(\Omega)$, there exists, according to Theorem 2.1, $u_\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ which is a solution of the equation:

(1) $\quad D_t u_\varepsilon - \varepsilon D_t^2 u_\varepsilon - D_x^2 u_\varepsilon - D_y^2 u_\varepsilon = f$

Since u_ε satisfies the estimate $\|u_\varepsilon\|_{H^{1,2}(\Omega)} \leqslant C\,\|f\|$ (see Propo. 2.2), where C does not depend on ε, there exists a subsequence u_{ε_j} (weakly) convergent in $H^{1,2}(\Omega)$ to a function u, when ε_j tends to 0. Clearly, $u \in H^{1,2}(\Omega) \cap H_o^1(\Omega)$ (because each u_{ε_j} is in the same space).

Moreover, Propo. 2.2. also shows that $\varepsilon_j\, D_t^2 u_{\varepsilon_j}$ is (weakly) convergent to 0. Thus, putting $\varepsilon = \varepsilon_j$ in Equation (1) and making $\varepsilon_j \to 0$, we get:

$$D_t u - D_x^2 u - D_y^2 u = f\,,$$

where $u \in H^{1,2}(\Omega) \cap H_o^1(\Omega)$. This proves the existence of the solution of problem (P).

As for the uniqueness of the solution, if we assume that $D_t u - \Delta u = 0$, we deduce:

$$0 = (D_t u - \Delta\, u,\, u) = \|D_x u\|^2 + \|D_y u\|^2,$$

consequently $D_x u = D_y u = 0$, thus $D_x^2 u = D_y^2 u = 0$.

The equation $D_t u - \Delta\, u = 0$ leads to $D_t u = 0$.

The solution u is then a constant and $u|_\Gamma = 0$.

Hence, $u \equiv 0$.

3.2. Case two: Ω does not verify the condition (H).

In this case, Γ_N is a polygonal domain. We shall reduce the problem to the previous case by a suitable extension of the domain Ω.

Let A be a point of R^3 such that its first coordinate t_o verifies $t_o > T = \sup t$.

We project the point A (parallely to the time axis) on the plane of Γ_N. A is chosen such that its projection (on Γ_N) belongs to Γ_N.

Then, the cone C of vertex A and base Γ_N (its altitude is $h = t_o - T$) is convex. By choosing h small enough, it is always possible to make the polyhedron $\Omega^\circ = \Omega \cup C$ convex. So, Ω° is a polyhedron which satisfies the hypothesis (H).

In general case, it is suitable to express this idea in terms of convex envelopes.

Now, if f is given in $L^2(\Omega)$ we denote by $\bar{f}$ ($\in L^2(\Omega^\circ)$) the continuation of f by 0 in Ω°. Hence, according to section 3.1., there exists $u \in H^{1,2}(\Omega^\circ) \cap H_o^1(\Omega^\circ)$ solution of $D_t u - \Delta\, u = \bar{f}$. Therefore, the restriction $u|_\Omega$ of u is a solution of problem (P) in

Ω. This proves the existence of the solution. As regards the uniqueness, the proof is similar to the first case.

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صقالة حل معادلة الحرارة في ساحة شاذة من R^{n+1}

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المدرسة العليا للأساتذة ـ القبة القديمة
مدينة الجزائر ـ الجزائر

نبرهن على أن مؤثر الحرارة تشاكل (تقابل) من فضاء لسوبولاف في فضاء الدوال ذات المربعات القابلة للمكاملة، وذلك باعتبار شروط ديركليت علماً أن الساحة المعرفة عليها تلك الدوال ساحة شاذة متعددة السطوح.