

On Transitive (2,3,6) Groups of Degree $n \leq 23$

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ABSTRACT. In this paper we investigate transitive (2,3,6) subgroups of S_n , $6 \leq n \leq 23$. We find 13 such groups up to isomorphism, we point out their structure and locate them in S_n . Also we interpret some of them geometrically.

Many of the important finite groups are generated by two elements. For instance the groups $\langle x, y \mid x^2 = y^2 = (xy)^n = 1 \rangle$, $n \geq 2$ are the dihedral groups, the groups $\langle x, y \mid x^2 = y^3 = (xy)^m = 1 \rangle$, where $m = 3, 4, 5$ are A_4, S_4, A_5 respectively (Coxeter and Moser 1980).

It seems natural to consider the next case, namely finite groups generated by x and y such that $x^2 = y^3 = (xy)^6 = 1$. We refer to these as (2,3,6) groups unless otherwise mentioned.

Now let us fix some notation:

1. S_n is the symmetric group on n objects.
2. C_n is the cyclic group of order n .
3. Let G be a finite group and $g, u \in G$, then $u^g = g^{-1} u g$.

If $a \in G$, then $O(a)$ is the order of a in G . If H is a subgroup of G , we write it $H \leq G$, while if H is a normal subgroup we write it $H \triangleleft G$. $Z(G)$ denotes the center of G , G' is the commutator subgroup of G , and $O(G)$ is the number of elements of G . If G is generated by a and b , we write this as $G = \langle a, b \rangle$.

4. If G is the split extension of $H \triangleleft G$ by $K \triangleleft G$, then we denote this by $G = H \rtimes K$.

5. In order to avoid repetition, G will always denote a $(2,3,6)$ group generated by x and y such that $x^2=y^3=(xy)^6=1$. $O(x)$ always will be 2, $O(y)=3$, while $O(z)=6$, where $z=xy$.

We start with a general fact about $(2,3,6)$ groups.

Proposition 1

G is a metabelian group *i.e.* G' is abelian. Moreover, $G' = \langle \alpha, \beta \rangle$ where $\alpha = [x, y]$ and $\beta = [x, y^{-1}]$.

Proof

Let $H = \langle \alpha, \beta \rangle$. We first show that H is abelian. Consider

$$\begin{aligned} [\alpha, \beta] &= \alpha^{-1}\beta^{-1}\alpha\beta = y^{-1}xyxyxy^{-1}xxy^{-1}xyxyxy^{-1} \\ &= y^{-1}xyxyxyxyxyxy^{-1} \\ &= y^{-1}(xy)^5xy^{-1} \\ &= y^{-1}(y^{-1}x)xy^{-1}, \text{ since } (xy)^6 = 1 \\ &= yy^{-1} = 1. \end{aligned}$$

Now

$$\alpha^x = y^{-1}xyx = \alpha^{-1}, \beta^x = yxy^{-1}x = \beta^{-1},$$

$\alpha^y = y^{-1}xy^{-1}xy^{-1} = \beta\alpha^{-1}$ and $\beta^y = \alpha^{-1}$. This shows that $H \trianglelefteq G$. Since $[x, y] \in H$, then G/H is abelian; hence $G' \leq H$, and it is clear that $H \leq G'$. Thus $H = G'$ which completes the proof of the proposition.

Remark

Considering G inside S_n for some n , note that α and β have the same cycle structure because $\beta^y = \alpha^{-1}$.

Now fix z in S_n for $n \geq 6$ such that $O(z)=6$. By a solution of

$$xy = z, \tag{1}$$

we mean an ordered pair $(x, y) \in S_n \times S_n$ with $O(x)=2$, $O(y)=3$ and $xy=z$. Two solutions (x_1, y_1) and (x_2, y_2) of (1) are said to be *conjugate* if $x_2 = x_1^\sigma$, $y_2 = y_1^\sigma$ where $\sigma \in C(z)$ the centralizer of z in S_n . By two *distinct solutions* of (1), we mean two non-conjugate solutions.

Let (x,y) be a solution of (1); then

$$\begin{aligned} z^{-1} &= y^{-1}x = x(xy^{-1}x) \\ &= xy'. \end{aligned}$$

If $\delta \in S_n$ is such that $(z^{-1})^\delta = z$, then

$$z = x^\delta(y')^\delta.$$

The solution $[x^\delta, (y')^\delta]$ is called the *associated solution* of (x,y) . If $[x^\delta, (y')^\delta]$ happens to be conjugate to (x,y) , we say that (x,y) is *self-associated*.

We say that a solution (x,y) of (1) gives rise to the subgroup G in S_n if $G = \langle x, y \rangle$. A solution (x,y) of (1) in S_n is said to have the cycle structure $2^{\lambda_1} \cdot 1^{\lambda_2}; 3^{\mu_1} \cdot 1^{\mu_2}; 6^{\epsilon_1} \cdot 3^{\epsilon_2} \cdot 2^{\epsilon_3} \cdot 1^{\epsilon_4}$ if x has λ_1 2-cycles, λ_2 fixed points, y has μ_1 3-cycles, μ_2 fixed points, and z has ϵ_1 6-cycles, ϵ_2 3-cycles, ϵ_3 2-cycles and ϵ_4 fixed points.

In the next proposition we list distinct solutions of (1) for all possible cycle structures in S_n for $6 \leq n \leq 23$. In order to avoid repetition in our list, by a solution of (1) we mean a solution giving rise to a transitive subgroup of S_n .

Proposition 2

The distinct solutions of (1) in S_n , $6 \leq n \leq 23$ are as follows, numbered according to their cycle structure.

In S_6 :

Fix z to be $(0,1,2,3,4,5)$

I $2^3; 3^2; 6$

$$x = (0,3)(1,4)(2,5)$$

II $2^3; 3 \cdot 1^3; 6$

$$x = (0,1)(2,3)(4,5)$$

III $2 \cdot 1^4; 3^2; 6$

$$x = (2,5) \underline{0} \underline{1} \underline{3} \underline{4}$$

In S_7 :

Fix z to be $(0,1,2,3,4,5) \underline{6}$

I $2^3 \cdot 1; 3^2 \cdot 1; 6 \cdot 1$

$$x = (1,3)(2,6)(4,5) \underline{0}$$

Associated solution is

$$x = (1,2)(3,5)(4,6) \underline{0}$$

In S_8 :

Fix z to be $(0,1,2,3,4,5)(6,7)$

$$\begin{aligned} \text{I} \quad & 2^4; 3^2.1^2; 6.2 \\ & x=(0,6)(1,2)(3,7)(4,5) \end{aligned}$$

In S_9 :

For case I fix z to be $(0,1,2,3,4,5)(6,7,8)$

$$\begin{aligned} \text{I} \quad & 2^3.1^3; 3^3; 6.3 \\ \text{I(i)} \quad & x=(2,5)(3,7)(4,8) \underline{0} \underline{1} \underline{6} \\ \text{I(ii)} \quad & x=(1,6)(3,8)(5,7) \underline{0} \underline{2} \underline{4} \\ & \text{For case II fix } z \text{ to be } (0,1,2,3,4,5)(6,7) \underline{8} \\ \text{II} \quad & 2^4.1; 3^3; 6.2.1 \\ & x=(1,7)(2,4)(3,8)(5,6) \underline{0} \end{aligned}$$

In S_{12} :

For cases I and II fix z to be $(0,1,2,3,4,5)(6,7,8,9,10,11)$

$$\begin{aligned} \text{I} \quad & 2^6; 3^4; 6^2 \\ & x=(0,3)(1,6)(2,7)(4,9)(5,10)(8,11) \\ \text{II} \quad & 2^4.1^4; 3^4; 6^2 \\ \text{II(i)} \quad & x=(2,5)(3,9)(4,10)(8,11) \underline{0} \underline{1} \underline{6} \underline{7} \\ \text{II(ii)} \quad & x=(1,10)(3,9)(4,7)(5,11) \underline{0} \underline{2} \underline{6} \underline{8} \\ \text{II(iii)} \quad & x=(1,10)(2,8)(4,7)(5,11) \underline{0} \underline{3} \underline{6} \underline{9} \\ & \text{For case III fix } z \text{ to be } (0,1,2,3,4,5)(6,7,8)(9,10) \underline{11} \\ \text{III} \quad & 2^6; 3^4; 6.3.2.1 \\ & x=(0,2)(1,11)(3,8)(4,10)(5,6)(7,9) \\ & \text{For case IV fix } z \text{ to be } (0,1,2,3,4,5)(6,7,8)(9,10,11) \\ \text{IV} \quad & 2^5.1^2; 3^4; 6.3^2 \\ \text{IV(i)} \quad & x=(1,6)(2,11)(4,10)(5,7)(8,9) \underline{0} \underline{3} \\ \text{IV(ii)} \quad & x=(0,7)(1,8)(2,5)(3,10)(4,11) \underline{6} \underline{9} \end{aligned}$$

In S_{13} :

Fix z to be $(0,1,2,3,4,5)(6,7,8,9,10,11) \underline{12}$

$$\begin{aligned} \text{I} \quad & 2^6.1; 3^4.1; 6^2.1 \\ & x=(0,1)(2,6)(4,11)(5,7)(8,10)(9,12) \underline{3} \end{aligned}$$

Associated solution is

$$x=(0,1)(2,6)(3,8)(4,7)(9,11)(10,12) \underline{5}$$

In S_{14} :

Fix z to be $(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13)$

I $2^7; 3^4.1^2; 6^2.2$

$$x=(0,1)(2,6)(3,9)(4,12)(5,7)(8,13)(10,11)$$

Associated solution is

$$x=(0,1)(2,6)(3,12)(4,10)(5,7)(8,9)(11,13)$$

In S_{15} :

Fix z to be $(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14)$

I $2^6.1^3; 3^5; 6^2.3$

I(i) $x=(2,5)(3,6)(4,7)(8,11)(9,13)(10,14) \underline{0} \underline{1} \underline{12}$

I(ii) $x=(1,9)(3,8)(4,12)(5,10)(7,13)(11,14) \underline{0} \underline{2} \underline{6}$

In S_{16} :

Fix z to be $(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14) \underline{15}$

I $2^8; 3^5.1; 6^2.3.1$

$$x=(0,2)(1,15)(3,6)(4,12)(5,7)(8,14)(9,10)(11,13)$$

In S_{18} :

For cases, I,II and III fix z to be

$z = (0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14,15,16,17)$

I $2^9; 3^6; 6^3$

I(i) $x=(0,6)(1,13)(2,8)(3,15)(4,10)(5,17)(7,16)(9,12)(11,14)$

I(ii) $x=(0,11)(1,6)(2,5)(3,17)(4,12)(7,10)(8,14)(9,15)(13,16)$

II $2^9; 3^5.1^3; 6^3$

$$x=(0,17)(1,2)(3,16)(4,9)(5,6)(7,8)(10,15)(11,12)(13,14)$$

III $2^7.1^4; 3^6; 6^3$

III(i) $x=(2,5)(3,13)(4,14)(8,11)(9,16)(10,17)(12,15)$

$$\underline{0} \underline{1} \underline{6} \underline{7}$$

III(ii) $x=(1,13)(3,12)(4,10)(5,14)(7,16)(9,15)(11,17)$

$$\underline{0} \underline{2} \underline{6} \underline{8}$$

$$\text{III(iii)} \quad x=(1,8)(2,15)(4,14)(5,9)(7,16)(10,13)(11,17)$$

$$\underline{0} \quad \underline{3} \quad \underline{6} \quad \underline{12}$$

Associated solution to III(iii) is

$$x=(1,9)(2,16)(4,15)(5,10)(7,13)(8,17)(11,14)$$

$$\underline{0} \quad \underline{3} \quad \underline{6} \quad \underline{12}$$

For case IV fix z to be

$$(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14)(15,16,17)$$

$$\text{IV} \quad 2^8.1^2; 3^6; 6^2.3^2$$

$$\text{IV(i)} \quad x=(1,7)(2,11)(3,17)(4,12)(5,8)(9,14)(10,15)(13,16)$$

$$\underline{0} \quad \underline{6}$$

Associated solution to IV(i) is

$$x=(1,10)(2,12)(3,16)(4,7)(5,11)(8,15)(9,13)(14,17)$$

$$\underline{0} \quad \underline{6}$$

$$\text{IV(ii)} \quad x=(1,14)(2,8)(3,17)(4,10)(5,12)(7,15)(9,13)(11,16)$$

$$\underline{0} \quad \underline{6}$$

$$\text{IV(iii)} \quad x=(0,14)(1,4)(2,8)(3,9)(5,13)(6,17)(7,10)(11,16)$$

$$\underline{12} \quad \underline{15}$$

For case V fix z to be

$$(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13)(14,15)(16,17)$$

$$\text{V} \quad 2^9; 3^6; 6^2.2^3$$

$$x=(0,13)(1,10)(2,15)(3,8)(4,16)(5,6)(7,17)(9,14)(11,12)$$

In S_{19} :

Fix z to be

$$(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14,15,16,17) \quad \underline{18}$$

$$\text{I} \quad 2^9.1; 3^6.1; 6^3.1$$

$$x=(1,12)(2,11)(3,7)(4,16)(5,13)(6,17)(8,10)(9,18)(14,15) \quad \underline{0}$$

Associated solution is

$$x=(1,12)(2,15)(3,11)(4,7)(5,13)(6,14)(8,10)(9,18)(16,17) \quad \underline{0}$$

In S_{21} :

For case I fix z to be

$(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14,15,16,17)(18,19) \underline{20}$

I $2^{10}.1; 3^7; 6^3.2.1$

$x=(1,11)(2,17)(3,8)(4,18)(5,6)(7,19)(9,16)(10,12)(13,15)(14,20) \underline{0}$

Associated solution is

$x=(1,11)(2,18)(3,9)(4,17)(5,6)(7,16)(8,12)(10,19)(13,15)(14,20) \underline{0}$

For case II fix z to be

$(0,1,2,3,4,5)(6,7,8,9,10,11)(12,13,14,15,16,17)(18,19,20)$

II $2^9.1^3; 3^7; 6^3.3.$

II(i) $x=(2,5)(3,11)(4,6)(7,10)(8,17)(9,12)(13,16)(14,19)(15,20)$

$\underline{0} \underline{1} \underline{18}$

II(ii) $x=(1,14)(2,7)(3,11)(4,18)(5,15)(8,13)(9,17)(10,19)(16,20)$

$\underline{0} \underline{6} \underline{12}$

Associated solution to II(ii) is

$x=(1,15)(2,18)(3,7)(4,11)(5,16)(8,20)(9,13)(10,17)(14,19)$

$\underline{0} \underline{6} \underline{12}$

There are no solutions in S_n for $n=10,20$, and 22 . Furthermore there are no solutions for $n \equiv 5 \pmod{6}$. Moreover, all the above solutions without the exceptions mentioned are self-associated.

Proof

We use inspection to get the above solutions. Also we make use of Theorem 2 in Singerman (1970) which asserts in our case that if a solution (x,y) of (1) which gives rise to a transitive subgroup of S_n has the cycle structure $2^{\lambda_1}.1^{\lambda_2}; 3^{\mu_1}.1^{\mu_2}; 6^{\epsilon_1}.3^{\epsilon_2}.2^{\epsilon_3}.1^{\epsilon_4}$, then there is an integer $g \geq 0$ such that

$$12g - 12 + 3\lambda_2 + 4\mu_2 + 3\epsilon_2 + 4\epsilon_3 + 5\epsilon_4 = 0 \quad (2)$$

This provides us with a necessary condition for the existence of solution (1). However, this condition is not sufficient as can be seen in S_{10} , where there are no solutions of (1) but the cycle structure $2^5; 3^3.1; 6.2^2$ satisfies (2) with $g=0$.

If $n \equiv 5 \pmod{6}$, then $\lambda_2 \geq 1$ and $\mu_2 \geq 2$. This does not leave enough room for ϵ_2, ϵ_3 and ϵ_4 to keep $g \geq 0$. Hence there are no solutions of (1) in S_n for $n \equiv 5 \pmod{6}$.

Direct computation shows that the groups whose generators are mentioned in Proposition 2 all have the structure $G \cong G' \rtimes C_6$, where $C_6 = \langle z \rangle$. The following Table reveals the structure of G , and we will refer to it in the next proposition:

Table 1.

| Degree | Case | Structure of G' | $Z(G)$ | $O(G)$ |
|--------|----------|-------------------|-----------------------|--------|
| 6 | I | Trivial | $\langle z \rangle$ | 6 |
| | II | C_3 | $\langle z^2 \rangle$ | 18 |
| | III | $C_2 \times C_2$ | $\langle z^3 \rangle$ | 24 |
| 7 | I | C_7 | $\langle 1 \rangle$ | 42 |
| 8 | I | $C_2 \times C_2$ | $\langle z^3 \rangle$ | 24 |
| 9 | I(i) | $C_3 \times C_3$ | $\langle 1 \rangle$ | 54 |
| | I(ii) | C_3 | $\langle z^2 \rangle$ | 18 |
| | II | $C_3 \times C_3$ | $\langle 1 \rangle$ | 54 |
| 12 | I | $C_2 \times C_2$ | $\langle z^3 \rangle$ | 24 |
| | II(i) | $C_4 \times C_4$ | $\langle 1 \rangle$ | 96 |
| | II(ii) | $C_4 \times C_4$ | $\langle 1 \rangle$ | 96 |
| | II(iii) | $C_2 \times C_2$ | $\langle z^3 \rangle$ | 24 |
| | III | $C_2 \times C_6$ | $\langle 1 \rangle$ | 72 |
| | IV(i) | $C_4 \times C_4$ | $\langle 1 \rangle$ | 96 |
| | IV(ii) | $C_4 \times C_4$ | $\langle 1 \rangle$ | 96 |
| 13 | I | C_{13} | $\langle 1 \rangle$ | 78 |
| 14 | I | C_7 | $\langle 1 \rangle$ | 42 |
| 15 | I(i) | $C_5 \times C_5$ | $\langle 1 \rangle$ | 150 |
| | I(ii) | $C_5 \times C_5$ | $\langle 1 \rangle$ | 150 |
| 16 | I | $C_4 \times C_4$ | $\langle 1 \rangle$ | 96 |
| 18 | I(i) | C_3 | $\langle z^2 \rangle$ | 18 |
| | I(ii) | $C_3 \times C_3$ | $\langle 1 \rangle$ | 54 |
| | II | $C_3 \times C_3$ | $\langle 1 \rangle$ | 54 |
| | III(i) | $C_6 \times C_6$ | $\langle 1 \rangle$ | 216 |
| | III(ii) | $C_2 \times C_6$ | $\langle 1 \rangle$ | 72 |
| | III(iii) | $C_6 \times C_6$ | $\langle 1 \rangle$ | 216 |
| | IV(i) | $C_6 \times C_6$ | $\langle 1 \rangle$ | 216 |
| | IV(ii) | $C_2 \times C_6$ | $\langle 1 \rangle$ | 72 |
| | IV(iii) | $C_6 \times C_6$ | $\langle 1 \rangle$ | 216 |
| | V | $C_3 \times C_3$ | $\langle 1 \rangle$ | 54 |
| 19 | I | C_{19} | $\langle 1 \rangle$ | 114 |
| 21 | I | C_{21} | $\langle 1 \rangle$ | 126 |
| | II(i) | $C_7 \times C_7$ | $\langle 1 \rangle$ | 294 |
| | II(ii) | C_7 | $\langle 1 \rangle$ | 42 |

Proposition 3

Any two (2,3,6) groups in S_n , $6 \leq n \leq 23$ of the same order are isomorphic. Hence there are at least 13 non-isomorphic groups in S_n , $6 \leq n \leq 23$.

Proof

For $6 \leq n \leq 23$ we mentioned that $G \cong G' \rtimes C_6$, where $C_6 = \langle z \rangle$. From Proposition 1, $G' = \langle [x,y], [x,y^{-1}] \rangle$ and $[x,y]^z = [x,y][x,y^{-1}]^{-1}$, $[x,y^{-1}]^z = [x,y]$ for all G . Therefore the statement of the proposition follows from table 1.

We emphasize the distribution of the isomorphism types of the (2,3,6) groups on S_n , $6 \leq n \leq 23$ in the following Table.

Table 2.

| Degree | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---------------|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| No. of groups | 3 | 1 | 1 | 2 | 0 | 0 | 3 | 1 | 1 | 1 | 1 | 0 | 4 | 1 | 0 | 3 | 0 | 0 |

Remark

The only primitive (2,3,6) groups in S_n , $6 \leq n \leq 23$, are the group of order 42 in S_7 , the group of order 78 in S_{13} and the group of order 114 in S_{19} .

Now we prove the following proposition that holds for all (2,3,6) groups.

Proposition 4

All (2,3,6) groups in S_n , $n \geq 6$ have the form $G \cong G' \rtimes C_6$, where $C_6 = \langle z \rangle$.

Proof

By Proposition 1, $G' = \langle \alpha, \beta \rangle$, where $\alpha = [x,y]$ and $\beta = [x,y^{-1}]$. If $x \in G'$, then since G' is abelian, $\alpha^x = \alpha$, $\beta^x = \beta$. But $\alpha^x = \alpha^{-1}$, $\beta^x = \beta^{-1}$, thus $O(G') \leq 4$ and $O(G) \leq 12$. From Table 1 the only (2,3,6) group G with $O(G) \leq 12$ is C_6 where G' is trivial. Hence $x \notin G'$.

If $y \in G'$, then since $\alpha^y = \beta \alpha^{-1}$ and $\beta^y = \alpha^{-1}$, we have $\alpha = \beta \alpha^{-1}$ and $\beta = \alpha^{-1}$. This implies that $\alpha^3 = 1$, thus $O(G') \leq 3$ and $O(G) \leq 6$. But the only case where $O(G) \leq 6$ is case I in S_6 in which G' is trivial, a contradiction. Therefore $G \cong G' \rtimes C_6$, where $C_6 = \langle z \rangle$, $z = xy$.

We find it interesting to point out a geometric interpretation of some of the solutions obtained in Proposition 2.

Coxeter and Moser (1980) showed that the infinite $(2,3,6)$ group generated by x and y with the presentation

$$x^2 = y^3 = (xy)^6 = 1$$

is the rotation group of the regular tessellations (or maps) of type $\{3,6\}$, $\{6,3\}$, and the quasi-regular $\{\frac{3}{6}\}$ tessellation of the Euclidean plane. The type $\{3,6\}$ is the lattice of equilateral triangles filling the plane; $\{6,3\}$ is the dual tessellation of regular hexagons. The midpoints of the edges of either type are the vertices of the quasi-regular $\{\frac{3}{6}\}$ tessellation, whose faces are alternate triangles and hexagons. The element x corresponds to a half-turn about a vertex of $\{\frac{3}{6}\}$, y is a rotation of period 3 about a vertex of $\{6,3\}$, while xy is a rotation of period 6 about a vertex of $\{3,6\}$. The elements $\alpha = [x,y]$, $\beta = [x,y^{-1}]$ correspond to unit translations in two different directions.

Now to link these ideas with the finite $(2,3,6)$ groups considered in our paper we proceed as follows: For any two non-negative integers b and c , not both zero, we denote by $(2,3,6)_{b,c}$ the group whose presentation is

$$x^2 = y^3 = (xy)^6 = [x,y]^b [x,y^{-1}]^c = 1 \quad (3)$$

This group is finite of order $6T$, where

$$T = b^2 + bc + c^2.$$

$(2,3,6)_{b,c}$ can be interpreted as the rotation group of the three finite maps

$$\{3,6\}_{b,c}, \{6,3\}_{b,c}, \{\frac{3}{6}\}_{b,c}$$

on the torus. Consequently, if T is not too small, the group $(2,3,6)_{b,c}$ will have representations of degree, T , $2T$ and $3T$ on the vertices of the maps $\{3,6\}_{b,c}$, $\{6,3\}_{b,c}$, and $\{\frac{3}{6}\}_{b,c}$ respectively. We will demonstrate the geometric representation of our finite $(2,3,6)$ groups by giving an example of each of the three types of representation:

1. For the case II in S_6 , the group has the presentation as in (3) with $b = c = 1$ which can be verified easily, hence $T = 3$. It has a representation of degree $2T = 6$ on the vertices of the $\{6,3\}_{1,1}$ map below:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | | 2 | 3 | | 2 | 3 | | 2 |
| | 0 | 5 | | 0 | 5 | | 0 | 5 |
| 1 | | 4 | 1 | | 4 | 1 | | 4 |
| | 2 | 3 | | 2 | 3 | | 2 | 3 |
| 5 | | 0 | 5 | | 0 | 5 | | 0 |

where x is the half-turn about the midpoint between 0 and 1, y is the rotation through 120° about the point 1, and $z = xy$ is the rotation through 60° about the center of the hexagon 0 1 2 3 4 5.

2. The representation of the same group above, acting on the vertices of the map $\{\frac{3}{6}\}_{1,1}$ is of degree $3T = 9$. It corresponds to the case I(ii) in S_9 . It is as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 0 | 5 | 8 | 2 | 3 | 7 | 0 |
| | 1 | | 4 | | 6 | | 1 |
| 8 | 2 | 3 | 7 | 0 | 5 | 8 | 2 |
| 4 | | 6 | | 1 | | 4 | 6 |
| 7 | 0 | 5 | 8 | 2 | 3 | 7 | 0 |

Here x is the half-turn about the vertex 0, y is the rotation through 120° about the center of the triangle 0 1 7, and $z = xy$ is the rotation through 60° about the center of an adjacent hexagon.

3. For the case I in S_7 , where $x = (1,3)(2,6)(4,5)_0$. We have $b=2$, $c=1$, hence $T = 7$. The group acts on the vertices of the map of type $\{3,6\}_{2,1}$ below

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 6 | 4 | 0 | 5 | 2 | 3 | 1 |
| 5 | 2 | 3 | 1 | 6 | 4 | 0 | 5 |
| 4 | 0 | 5 | 2 | 3 | 1 | 6 | 4 |
| 3 | 1 | 6 | 4 | 0 | 5 | 2 | 3 |
| 5 | 2 | 3 | 1 | 6 | 4 | 0 | 5 |

where x is a half-turn about the midpoint between 1 and 3, y is the rotation through 120° about the center of the triangle 0 1 4 and $z = xy$ is the rotation through 60° about the vertex 6.

We would like to mention that the above geometric interpretation is not the only one for finite (2,3,6) groups. For example the group of order 24 in S_6 case III can not be interpreted in the above fashion; however, we can think of it along the following lines. It can be verified that this group is the pyritohedral group (Coxeter and Moser 1980 p. 39). Hence this group has a representation of degree 6 on the vertices of the regular octahedron. Here x is the reflection that interchanges two opposite vertices and y is the rotation about the line joining the midpoints of two opposite faces.

More research remains to be carried out along these lines. For instance, finding a geometric interpretation for the group in S_{12} , case I. Also interpreting the group of order 150 in S_{15} , cases I(i) and I(ii), where $T=25$.

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حول الزمر المتعدية (2,3,6) من الدرجة $23 \geq n$

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في بحثنا هذا توصلنا إلى أنه يوجد ثلاث عشرة زمرة متعدية وغير متشاكله من النوع (2,3,6)، تظهر كزمر جزئية من زمر التظاهر S_n ، $23 \geq n \geq 6$. وقد حددنا شكل كل زمرة وموقعها في S_n كما أعطينا تفسيراً هندسياً لبعضها.