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Sizable is Equivalent to Submetrizable

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ABSTRACT. The following result is proved in this paper:

Let X be a topological space.

Then the following are equivalent:

a) X is subsizable.
b) X is sizable.
c) X is submetrizable.

Several results are obtained as corollaries of the above theorem. These corollaries show the powerfulness of this theorem and show that the further study of sizable spaces is unwarranted.

1. Introduction

Let (X, τ) be a topological space and let $\mathcal{B}$ be a base for T. Let $L:(XxX) \cup \mathcal{B} \rightarrow (o, \infty)$ be a function. Then L is called a size function for $(X, T(\mathcal{B}))$ if it satisfies the following conditions:

L_1) $L(x,y) = O$ if and only if $x = y$.

L_2) $L(x,y) = L(y,x)$ for all $x,y \in X$

L_3) For any $x \in X$, for any open set U_x containing x, and for any positive real number r, there exists a basic open set $V_{x,r} \in \mathcal{B}$ such that

$x \in V_{x,r} \subseteq U_x$ and $L(V_{x,r}) < r$.

L_4) For any $V,V' \in \mathcal{B}$ and for any $x,x' \in V$; $y,y' \in V'$, we have

$$L(x',y') \leq L(x,y) + L(V) + L(V').$$

A space (X,T) is called a sizable space if the topology T on X has a base $\mathcal{B}$ and an associated size function L. This space will be denoted by (X, T($\mathcal{B}$),L).

A topological space is called submetrizable (resp. subsizable) if it is mapped onto some metrizable (resp. sizable) space by a continuous, one- to- one map.

The following theorem characterizes size functions on a given topological space (X,T). For the proof, consult (Fora 1983).

1.1 Theorem

i) Let (X,T ($\mathcal{B}$), L) be a sizable space. Let V ε $\mathcal{B}$ be a basic open set. Then for any a,b ε V and for any xεX, we have

$$L(x,b) \leq L(x,a) + L(V).$$

ii) Let (X,T ($\mathcal{B}$)) be a topological space and let $L: X x X \cup \mathcal{B} \rightarrow [o, \infty)$ be a function satisfying (L_1), (L_2), (L_3) and the following condition:

$L_4)'$ For any V ε $\mathcal{B}$ and any a,b ε V, x ε X, we have

$$L(x,b) \leq L(x,a) + L(V).$$

Then L is a size function for (X,T ($\mathcal{B}$)).

The following two theorems give some properties of sizable spaces (see Fora 1983 and Fora 1984 b).

1.2 Theorem

a) The product of two sizable spaces is sizable.
b) "Being sizable" is a hereditary property.
c) "Being sizable" is a topological property.

1.3 Theorem

a) Every metric space is sizable.
b) Every point in any sizable space is a G_δ – set.

2. Main Result

We shall start this section by the following theorem.

2.1 Theorem

Let (X,T) be a topological space.

Then the following are equivalent.

(1) X is subsizable.

(2) X is sizable.

(3) X is submetrizable.

Proof

(1) implies (2): Let (X,T) be a subsizable space. Then there exists a sizable space (Y,T_Y) and there exists a one-to-one map $f:X \to Y$ which is continuous. Since (Y,T_Y) is sizable, there exists a base $\mathcal{B}_Y$ for T_Y and there exists a size function L_Y for $(Y,T(\mathcal{B}_Y))$.

Define $\mathcal{B}$ as follows:

$$\mathcal{B} = \{G \cap f^{-1}(B): G \in T \text{ and } B \in \mathcal{B}_Y\} - \{\phi\}.$$

Then $\mathcal{B}$ is a base for T because f is continuous and $\mathcal{B}_Y$ is a base for T_Y.

Define L: $(X x X) \cup \mathcal{B} \longrightarrow [0, \infty)$ as follows:

$$L(x,x') = L_Y(f(x), f(x')), \; x,x' \in X; \text{ and}$$
$$L(U) = \inf \{L_Y(B) : U = G \cap f^{-1}(B), G \in T \text{ and } B \in \mathcal{B}_Y\}, \; U \in \mathcal{B}.$$

Then L is a size function for $(X,T(\mathcal{B}))$ because: (L_1) and (L_2) are easy to prove because L_Y is a size function and f is a one-to-one map.

To prove (L_4), let $V,V' \in \mathcal{B}$ and let $x,x' \in V; y,y' \in V'$. Let r be any positive real number. Then, from the definition of L, there exist $G \in T$, $G' \in T$, $B \in \mathcal{B}_Y$, $B' \in \mathcal{B}_Y$ such that

$$V = G \cap f^{-1}(B), \; V' = G' \cap f^{-1}(B') \text{ and}$$

$$L(V) \le L_Y(B) < L(V) + \tfrac{1}{2} r, \; L(V') \le L_Y(B') < L(V'); + \tfrac{1}{2} r.$$

Since $x,x' \in V; y,y' \in V'$;, therefore $f(x), f(x') \in B$ and $f(y),f(y') \in B'$. Applying (L_4) for L_Y, we get

$$L(x',y') = L_Y(f(x'), f(y')) \le L_Y(f(x), f(y)) + L_Y(B) + L_Y(B') < L(x,y)+L(V)+L(V')+r.$$

Since r is an arbitrary small positive real number, therefore $L(x',y') \le L(x,y) + L(V)+L(V')$ and hence (L_4) is proved.

To prove (L_3), let $x \in X$ and let U_x be any open set in X containing x. Let r be any positive real number. Let $B \in \mathcal{B}_Y$ be any basic open set such that $f(x) \in B \subseteq Y$ and

$L_Y(B) < r$. Then $x \varepsilon V = U_x \cap f^{-1}(B) \varepsilon \mathcal{B}$ and $L(V) \leq L_Y(B) < r$. This completes the proof of the implication.

(2) implies (3): Let $(X, T(\mathcal{B}), L)$ be a sizablespace. We may assume that $L \leq 1$ and $X \varepsilon \mathcal{B}$ (see Fora 1983).

The following definition is needed: A basic chain from $x \varepsilon X$ into $y \varepsilon X$ is a finite sequence of basic open sets $\{V_1, V_2, \ldots V_n\}$ such that $x \varepsilon V_1$, $y \varepsilon V_n$ and $V_i \cap V_{i+1} \neq \phi$ for $i=1,2, \ldots, n-1$. Notice that basic chains always exist between points in X because $X \varepsilon \mathcal{B}$. Define $d: X x X \rightarrow R$ as follows:

$$d(x,y) = \inf \{ \sum_{i=1}^{n} L(V_i) : (V_i)_{i=1}^{n} \text{ is a basic chain from } x \text{ into } y\};\ x,y \ \varepsilon\ X.$$

Then it is clear that $d(x,y) = d(y,x)$ for all $x,y \ \varepsilon\ X$. To prove the triangle inequality of d, let $x,y,z \ \varepsilon\ X$ and let r be any positive real number. From the definition of d, there exist two basic chains:

$(V_i)_{i=1}^{n}$ from x into z, and $(U_i)_{i=1}^{m}$ from z into y such that

$$d(x,z) \leq \sum_{i=1}^{n} L(V_i) < d(x,z) + \tfrac{1}{2} r \text{ and}$$

$$d(y,z) \leq \sum_{i=1}^{m} L(U_i) < d(y,z) + \tfrac{1}{2} r.$$

Consequently, we have

$$d(x,z)+d(y,z) \leq \sum_{i=1}^{m} L(U_i) + \sum_{i=1}^{n} L(V_i) < d(x,z)+d(y,z) + r.$$

Since $\{V_1, V_2, \ldots V_n, U_1, U_2, \ldots U_m\}$ is clearly a basic chain from x into y, therefore

$$d(x,y) \leq \sum_{i=1}^{m} L(U_i) + \sum_{i=1}^{n} L(V_i)$$

Hence, we get $d(x,y) < d(x,z) + d(y,z) + r$ for every $r > O$.

Consequently, we have $d(x,y) \leq d(x,z) + d(y,z)$.

It is clear that (L_3) implies that $d(x,x)=0$ for every $x \ \varepsilon X$.

The step "$d(x,y) = 0$ implies $x=y$" can be proved from (L_4) and the use of the following remark.

2.2 Remark

If $(V_i)_{i=1}^n$ is a chain from x to y then $L(x,y) \leq 2\sum_{i=1}^{n} L(V_i)$.

Proof

Let $c_i \varepsilon V_i \cap V_{i+1}$ $(1 \leq i < n)$. Then we have the following three cases:

case $n=1$. For $x', y' \varepsilon V_1$, take $(x=y'=x')$ by (L_4) we have
$L(x,y) \leq L(x,x) + L(V_1) + L(V_1) = 2L(V_1)$.

case $n=2$, $c_1, x \varepsilon V_1$ and $c_1, y \varepsilon V_2$, so by (L_4) we have
$L(x,y) \leq L(c_1,c_1) + L(V_1) + L(V_2)) \leq 2(L(V_1) + L(V_2))$.

case $n \geq 3$ (by induction), c_1, $x \varepsilon V_1$ and c_{n-1}, $y \varepsilon V_n$, so by (L_4) we have

$$L(x,y) \leq L(c_1,c_{n-1}) + L(V_1) + L(V_n)$$
$$\leq 2\sum_{i=2}^{n-1} L(V_i) + L(V_1) + (L(V_n) \leq 2\sum_{i=1}^{n} L(V_i).$$

To complete the proof, that d is a metric on X, let $d(x,y)=0$. Take any $r>0$. Then there exists a basic chain $(V_i)_{i=1}^n$ from x into y such that $0 \leq \sum_{i=1}^{n} L(V_i) < \frac{1}{2}r$. Using the above remark, we get $L(x,y)<r$ for every $r>0$. Consequently, we have $L(x,y)=0$ which implies that $x=y$.

To complete the proof of the implication, we are going to prove that $T(d) \subseteq T(\mathcal{B})$. To do this, let $y \varepsilon B_d(x,r)$ (the d-open sphere with center x and radius r), where $x \varepsilon X$ and $r>o$. Then $d(x,y)<r$. From the definition of d, there exists a basic chain $(V_i)_{i=1}^n$ from x into y such that $d(x,y) \sum_{i=1}^{n} L(V_i) < r$. We claim that $V_n \subseteq B_d(x,r)$.

To prove our claim, let $z \varepsilon V_n$. Then $(V_i)_{i=1}^n$ is a basic chain from x into z. Therefore $d(x,z) \leq \sum_{i=1}^{n} L(V_i) < r$. Thus $z \varepsilon B_d(x,r)$ and our claim is proved. Now, it is clear that the identity map

$i: (X,T(\mathcal{B})) \to (X,T(d))$, $i(x)=x, x \varepsilon X$, is continuous and a bijection.

Hence $(X,T(\mathcal{B}))$ is submetrizable.

(3) implies (1): the proof is obvious because of Theroem 1.3 (a).

3. Significance of the main result

This section is devoted for emphasizing the importance and the powerfulness of Theorem 2.1. As a conclusion of the results which will be stated in this section we shall see that the further study of sizable spaces is unwarranted.

Let us start this section by the following result.

3.1. Theorem

If X is a countably compact sizable space. Then X is metrizable.

Proof

Since X is sizable, therefore X is submetrizable. Hence there exists a metric space (Y,d) and there exists a bijective continuous map f: $X \rightarrow Y$. If F is a closed subset of X, then F is countably compact and therefore f(F) is also countably compact. Since Y is a metric space, therefore f(F) is a compact set and hence it is closed.

Consequently, the map f is a closed map and hence it is a homeomorphism. Thus X is metrizable.

We would like to point out that Theorem 3.1 was proved in (Fora 1984 b) by using a different complicated approach.

To prove the next result, we need the following lemma.

3.2 Lemma

Let I be an uncountable set of indices and let X_i be spaces containing more than one element for uncountably many i's. Then no point of the product space $X=\pi\{X_i:i\varepsilon I\}$ is a G_δ – set.

Proof

Let $\rho=(x_i)_{i\varepsilon I}$ be an element in the product space X. To prove that $\{\rho\}$ is not a G_δ–set, suppose on the contrary; then there exist basic open sets G_k ($k\varepsilon N$) in X such that $\{\rho\} = \bigcap_{k=1}^{\infty} G_k$. Since G_k is a basic open set in X, therefore there exist basic open sets $U_{\alpha i,k}$ in $X_{\alpha i,k}$ (i=1, ... n,(k)) such that

$$G_k = U_{\alpha 1,k} x \dots x\ U_{\alpha n(k),k}\ x\ \pi\ \{X_i\!:\ i \neq \alpha 1,k, \dots ,\alpha\ n(k),k\}.$$

Let $\alpha\varepsilon I-\{\alpha_{i,j}:i=1,2, \dots, n(j), j\varepsilon N\}$ be such that X_α contains at least two elements. Then there exist $x_{\alpha 1}\ \varepsilon\ X_\alpha$ and $x_{\alpha 2}\ \varepsilon\ X_\alpha$ such that $x_{\alpha 1} \neq x_{\alpha 2}$.

Let $y=(y_i)_{i\varepsilon I}$ and $z=(z_i)_{i\varepsilon I}$ be elements in X such that $y_i= z_i=x_i$ for $i\neq \alpha$ and $y_\alpha = x_{\alpha 1}$, $z_\alpha=x_{\alpha 2}$. Then it is clear that $\bigcap_{k=1}^{\infty} G_k$ contains the distinct points y and z, and this contradicts the fact that $\{\rho\}= \bigcap_{k=1}^{\infty} G_k$. This completes the proof of the lemma.

Now, we are ready to prove the next result.

3.3. Theorem

A nonempty product space $\pi\{X_i:i\varepsilon I\}$ is sizable if and only if each X_i is sizable and X_i is a single point for all but a countable set of indices.

Proof

($\longrightarrow$) Each X_i is homeomorphic to a subspace of the product and hence; according to Theorem 1.2 (b,c); sizable. Moreover, every point in the product space is a G_δ−set, if sizable, and thus can be at most a countable product (see Theorem 1.3(b) and Lemma 3.2).

($\longleftarrow$): Let X_1, X_2, ... be sizable spaces.

Then, by the use of Theorem 2.1, there exist metrizable spaces $Y_1,Y_2,\ldots$ and continuous bijective mappings $f_i:X_i \longrightarrow Y_i$, i εN. Define

$f: \pi_{i=1}^{\infty} X_i \longrightarrow \pi_{i=1}^{\infty} Y_i$ by

$f(x_1,x_2, \ldots) = (f_1(x_1), f_2(x_2), x_i\varepsilon X_i$. Then it is easy to check that f is a bijective map and continuous. Since $\pi_{i=1}^{\infty} Y_i$ is metrizable according to Theorem 22.3 in (Willard 1970, p.161), therefore $\pi_{i=1}^{\infty} X_i$ is a submetrizable space and hence, by Theorem 2.1, sizable.

Notice that Theorem 3.3 generalizes Theorem 2.4 in (Fora 1984 a), from finite products to arbitrary products.

As was shown in (Fora 1983), size functions need not be monotonic. The following theorem shows that every sizable space admits a monotonic size function.

3.4. Theorem

Let (X,T) be a sizable space. Then there exists a base $\mathcal{B}$ for T and a size function L for (X,T) satisfying the condition:

If U, V$\varepsilon\mathcal{B}$ and $U \subseteq V$, then $L(U)\leq L(V)$

Proof

Let (X,T) be a sizable space. Then, by the use of Theorem 2.1, there exists a metric space (Y,d) and a continuous bijective mapping $f:X \to Y$. Let $\mathcal{B}_Y$ denote the base for Y consisting of all open spheres in Y. Define $\mathcal{B}$ as follows:

$$\mathcal{B} = \{G \cap f^{-1}(B) : G \varepsilon T \text{ and } B \varepsilon \mathcal{B}_Y\} - \{\phi\}.$$

Then $\mathcal{B}$ is a base for T because f is continuous and $\mathcal{B}_Y$ is a base for Y.

Define $L : (X \times X) \cup \mathcal{B} \to [o, \infty)$ as follows:

$$L(x,x') = d(f(x), f(x')), \quad x,x' \varepsilon X; \text{ and}$$

$$L(U) = \inf \{\delta_d(B) : U = G \cap f^{-1}(B), G \varepsilon T \text{ and } B \varepsilon \mathcal{B}_Y\}, \quad U \varepsilon \mathcal{B}.$$

Then L is a size function for (X,T) (the proof is the same as in Theorem 2.1:(1) implies (2)) and moreover it is easy to check that $L(U) \leq L(V)$ whenever $U \subseteq V$; U, $V \varepsilon \mathcal{B}$.

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الفضـــاءات الحجميــة تكــافيء الفضــاءات المترية الجزئية

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لقد أثبتنا النتيجة التالية في هذا البحث:
لنفرض أن س فضاء ا توبولوجيا. فإن جميع ما يلي متكافئين:
أ) س فضاء حجمي جزئي.
ب) س فضاء حجمي.
جـ) س فضاء متركي جزئي.

٢ ـ يكـون فضاء الجـداء π { سـ$_{ن}$: ن ∋ ع } حجمياً إذا وفقط إذا كان كل فضاء سـ$_{ن}$ حجمياً وكـان سـ$_{ن}$ فضاءاً نقطياً (مكون من نقطة واحدة) لجميع ن المـوجودة في ع باستثناء عدد معدود منها.

٣ ـ إذا كان سـ فضاء مـتراص معدود وحجمي فـإنه يكـون متركي.

٤ ـ كـل فضاء حجمي سـ لـه قاعـدة بـ ودالـة حجميـة ق تحقق ما يلي:
إذا كان ل، ك ∋ بـ، ل ⊇ ك فإن ق (ل) ≥ ق (ك).