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ORIGINAL PAPER

F M Al-Sunni

Controller Design for Interval Plants

Abstract:We make use of the celebrated Kharitonov theorem to come up with a design procedure for the stabilization of uncertain systems in the parameters using low order controllers. The proposed design is based on classical design methods. A Non-Linear Programming (NLP) approach for the design of higher order controllers is also presented. We present our results and give illustrating examples.

Keywords: Controller, design, Classical methods, nonliner programming, Kharitonov theorem

تصميم متحكمات لأنظمة ذات فترة

فؤاد محمد السني

المستخلص: نستخدم نظرية خاريتانوف من أجل تصميم متحكمات لاستقرار أنظمة بها نسبة أخطاء. نستخدم الطرق التقليدية في تصميم المتحكمات ذات الحجم الصغير ونستخدم طريقة البرمجة الغير خطية للمتحكمات ذات الحجم الأكبر. نستعرض في هذه الورقة الطرق المقترحة ونقدم بعض الأمثلة.

كلمات مدخلية: متحكمات، تصميم، طرق تقليدية، برمجة غير خطية، خاريتانو، نظرية.

Introduction

The celebrated Kharitonov theorem (Kharitonov, 1978) has initiated a lot of work on the study of stability of interval plants. The reader is referred to the results reported in (Hernandez & Dormido, 1995), (Hernandez *et al.* 1998), (Djaferis, 1993), (Ghosh, 1985), and (Hollot & Yang, 1990), and the references therein. However, very few efforts have been reported in the areas of stabilizability and the design of controllers for interval plants. In (Ho *et al.* 1998), P, PI, and PID controllers are designed based on the stabilization of sixteen plants. $H\infty$ based methods for the stabilization of this class of systems are reported in (Bahattacharyya *et al.* 1993). Ghosh (1955), and Hollot *et al.* (1990) give conditions on the stabilizability of interval plants using low-order controllers.

It is the objective of this paper to develop a simple scheme for the design of low order controllers to stabilize interval plants. The stabilization of interval plants is achieved through the stabilization of four polynomials derived from the closed-loop characteristic equation of the interval plant. The stabilization of the four polynomials may be achieved by some classical controller design methods such as the root locus. The extension to the design of higher order controllers is also addressed.

In the following section, the problem we are addressing is stated. In Section 3, controller design procedure for pure gain, Lead-Lag and PI controllers is presented. Higher-order controllers are presented in Section 4, and an NLP based approach for the design is presented. Examples are presented in Section 5. We conclude the paper in Section 6.

F M Al-Sunni
Dept. of Systems Engineering
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
E-mail:ealsunni@ccse.kfupm.edu.sa

Background and Statement of the Problem

We are given the system

$$G(s,c) = \frac{Q(s,b)}{P(s,a)} \tag{1}$$

with

$$P(s,a) = a_0 + a_1 s + \ldots\ldots\ldots\ldots a_{n_p-1} s^{n_p-1} + a_{n_p} s^{n_p}$$
$$Q(s,b) = b_0 + b_1 s + \ldots\ldots\ldots\ldots b_{n_p-1} s^{n_p-1} + b_{n_p} s^{n_p}$$

and

$$\begin{aligned} a_i &\in [a_i, \bar{a}_i] \\ a_i &\in [\underline{b}_i, \bar{b}_i] \end{aligned} \tag{2}$$

$$c = \lfloor a_0\, a_1 \ldots\ldots a_{n_p}\; b_0\, b_1 \ldots\ldots b_{n_p} \rfloor.$$

The stability of the system in (1) can be tested by checking the stability of the following four Kharitonov polynomials (Kharitonov, 1978),

$$
\begin{aligned}
P_1(s) &= \underline{a}_0 + \underline{a}_1 s + \overline{a}_2 s^2 + \overline{a}_3 s^3 + \dots\dots\dots \\
P_2(s) &= \underline{a}_0 + \overline{a}_1 s + \overline{a}_2 s^2 + \underline{a}_3 s^3 + \dots\dots\dots \\
P_3(s) &= \overline{a}_0 + \underline{a}_1 s + \underline{a}_2 s^2 + \underline{a}_3 s^3 + \dots\dots\dots \\
P_4(s) &= \overline{a}_0 + \overline{a}_1 s + \underline{a}_2 s^2 + \underline{a}_3 s^3 + \dots\dots\dots
\end{aligned}
\tag{3}
$$

The problem we are addressing in this paper is the design of a controller of order n_c given by

$$
C(s) = \frac{N(s)}{D(s)} \tag{4}
$$

such that the system in (1) in stable for all values of $a_i's$ and $b_i's$ given by (2).

Several results related to the stabilization of interval plants have been reported in the literature. A lot of these results are based on the above Kharotonov polynomials, but few are addressing the design of controllers for such plants. In the following section we present the proposed design scheme for low-order controllers. The assumption we make is that the interval plant is stabilizable using the selected order for the controller.

Controller Design

The closed-loop system G_{cl} (*s*)with a plant given by (1) and a controller *C*(*s*) given by (4) is given by

$$
G_{cl}(s) = \frac{N(s)Q(s,b)}{N(s)Q(s,b)+D(s)P(s,a)} \tag{5}
$$

This system is stable for all $a_i's$ and $b_i's$ and according to (2) if the closed-loop characteristic polynomial

$$
\Delta_{cl}(s) = N(s)Q(s,b)+D(s)P(s,a) \tag{6}
$$

is stable for all $a_i's$ and $b_i's$

The design procedure we propose involves the following two steps. The first step is to write the closed-loop characteristic equation in the form.

$$
\Delta_{cl}(s) = \sum_{i=0}^{n_p+n_c} l_i s^i \tag{7}
$$

where l_i is a function of both the plant and the controller parameters. Now, one may produce the four Kharitonov polynomials given in (3). The next step is to select the controller parameters such that the four polynomials are stable. For low-order controllers, one may use the Routh-Hurwitz stability test, or the Root Locus method to come up with good controller parameters.

In the following sub-sections, pure gain, Lead-Lag, and PI controllers are designed based on this procedure.

Pure Gain Controllers

If the controller is a pure gain controller, then the closed-loop characteristic polynomial is given by

$$\Delta_{cl}(s)=l_0+l_1 s+..........+l_{n_p} s^{n_p}$$

and this can be written in the form (7) according to

$$\Delta_{cl}(s)=l_0+l_1 s+..........+l_{n_p} s^{n_p}$$

with $l_i=(a_i+b_i K)$. The four Kharitonov polynomials can be generated according to (3)

with $\underline{l}_i=(\underline{a}_i+\underline{b}_i K)$ and $\bar{l}_i=(\bar{a}_i+\bar{b}_i K)$ for positive K.

Corollary 1: Let the closed-loop characteristic polynomials be given by $\Delta_{cl}(s)$, let $\Delta^i_{cl}(s)$ for i = 1,2,3,4 be the four Kharitonov polynomials associated with. Write the four $\Delta^i_{cl}(s)$ as follows

$$\Delta^i_{cl}(s)=P_i^1(s)+P_i^1(s)K \tag{8}$$

Then the closed-loop system of the interval plant (1) with a pure gain controller is stable if and only if the following four virtual systems are stable for some K.

for $$G_i^v(s)=\frac{P_i^2(s)}{P_i^1} \text{ for } i=1,2,3,4 \tag{9}$$

Proof: If $G_i^v(s)$ are stable for i = 1, 2, 3, 4 for some gain K, then $P_i(s)$ are stable for i = 1, 2, 3, 4 for the same value of K. Using the result from the Kharitonov theorem, this results in the stability of the closed-loop characteristic equation. This proves the if direction for the corollary. If $\Delta_{cl}(s)$ is stable, then the four Kharitonov polynomials associated with this $\Delta_{cl}(s)$ are also stable. This leads to the stability of the four virtual systems $G_i^v(s)$.

A design scheme based on the root locus can be suggested which makes use of the above theorem. Four root loci can be generated for these four systems, and a value of K, which stabilizes all of them can be found. This K stabilizes the original system for all the given uncertainty.

This procedure was used to come up with a gain that will stabilize the interval plant used in (5). We obtained $K > 5.4721$. This matches the result obtained in (5). However, it was obtained with fewer computations, since sixteen plants need to be checked in (5).

Lead-Lag Controller

Now we present the result for the stabilization of (1) using a Lead-Lag controller which is given by

$$C(s)=K\frac{s+z}{s+p}$$

We assume $z > 0$, and $p > 0$ and $K > 0$.
The closed-loop characteristic polynomial with this controller is given by

$$\Delta_{cl}(s)=K(s+z)Q(s,b)+(s+p)P(s,a)$$

The closed-loop polynomials can be written in the form (7) according to

$$\Delta_{cl}(s)=l_1+l_1 s+.............+l_{n_p+1} s^{n_p+1} \tag{10}$$

with

$$
\begin{aligned}
l_0 &= pa_0 + zb_0K \\
l_i &= (a_i + a_{i-1}p) + (b_i + zb_{i-1}) \text{ for } i = 1,....,n_p \\
l_{n_p+1} &= K + 1
\end{aligned}
\tag{11}
$$

Corollary 2: For fixed values of z and p, the interval plant (1) is stabilizable by a lead-lag controller if and only if the following four virtual plants are stabilized by pure gain K,

$$
\begin{aligned}
G_1^v(s) &= \frac{z\,\underline{b}_0 + (\bar{b}_1 + z\,\bar{b}_0)s + (\bar{b}_2 + z\,\bar{b}_1)s^2 .... + s^{n_p}}{p\,\underline{a}_0 + (\underline{a}_1 + p\,\underline{a}_0)s + (\bar{a}_2 + p\,\bar{a}_1)s^2 .... + s^{n_p}} \\
G_2^v(s) &= \frac{z\,\underline{b}_0 + (\bar{b}_1 + z\,\bar{b}_0)s + (\bar{b}_2 + z\,\bar{b}_1)s^2 .... + s^{n_p}}{p\,\underline{a}_0 + (\bar{a}_1 + p\,\bar{a}_0)s + (\bar{a}_2 + p\,\bar{a}_1)s^2 .... + s^{n_p}} \\
G_3^v(s) &= \frac{z\,\bar{b}_0 + (\underline{b}_1 + z\,\underline{b}_0)s + (\underline{b}_2 + z\,\underline{b}_1)s^2 .... + s^{n_p}}{p\,\bar{a}_0 + (\underline{a}_1 + p\,\underline{a}_0)s + (\underline{a}_2 + p\,\underline{a}_1)s^2 .... + s^{n_p}} \\
G_4^v(s) &= \frac{z\,\bar{b}_0 + (\bar{b}_1 + z\,\bar{b}_0)s + (\underline{b}_2 + z\,\underline{b}_1)s^2 .... + s^{n_p}}{p\,\bar{a}_0 + (\bar{a}_1 + p\,\bar{a}_0)s + (\underline{a}_2 + p\,\underline{a}_1)s^2 .... + s^{n_p}}
\end{aligned}
\tag{12}
$$

Proof: Derive the Kharotonov polynomials, and write the closed-loop characteristic equation in the form (8). The argument used in the proof of the previous theorem applies.

One may get expressions for $\underline{l}_i$ and $\bar{l}_i$, as functions of the upper and the lower bounds for the parameters of the interval plant. Now, we re-arrange the polynomial parameters to write it in a form similar to the one in (8) which will result in the virtual plants given by (9). To stabilize the original systems for all parametric uncertainties, one has to select the controller parameters, *z, p* and *K* such that the four polynomials are stable. By examining the root locus for the pure gain controller, one may suggest stabilizing locations for the controller pole and zero. Then, we are left with the choice of *K*. This could be done along the same lines as suggested for the pure gain case.

PI Controller

The PI controller is very popular among control engineers. It is given by

$$
C(s) = K_p + \frac{K_i}{s} = K_p \frac{s + K_i/K_p}{s} \tag{13}
$$

This is in the lead-lag form with , and . It is possible to generate the closed-loop polynomial as given in (10) with the following parameters

$$
\begin{aligned}
l_0 &= zb_0K_p \\
l_i &= a_i + K_p(b_i + zb_{i-1}) \quad for \quad i = 1, ..., n_p \\
l_{n_p+1} &= K_p + 1.
\end{aligned}
\tag{14}
$$

Corollary 3: For a given location of the zero of the PI controller, the interval plant (1) is stabilizabled by a PI controller if and only if the four virtual plants similar to the ones given by (12) with $\pi = 0$, are stabilizable by pure gain K.

Proof: Same as for the previous theorem.
Starting with $z = 0$, one can generate the root locus for the four virtual systems. The root locus can be used to select a stabilizing location for the zero of the controller. After this, the controller gain,K_p, can be selected such that all the four systems are stable.

Higher-Order Controllers

In this section, we develop an NLP approach for the design of n^{th} order controller. Consider the system given in (1) and the proposed controller is a stable minimum-phase given by

$$C(s) = K\frac{s^{n_c} + b_{n_c-1}^c s^{n_c-1} \ldots\ldots\ldots + b_1^c s + b_0^c}{s^{n_c} + a_{n_c-1}^c s^{n_c-1} \ldots\ldots\ldots\ldots + a_1^c s + a_0^c} \tag{15}$$

The closed-loop characteristic equation for the system using the above controller is given by

$$\Delta_{cl}(s) = l_0 + l_1 s + \ldots\ldots\ldots\ldots + l_{n_p+n_c} s^{n_p+n_c} \tag{16}$$

with

$$l_0 = a_0 a_0^c + K b_0 b_0^c$$

$$l_j = \sum_{m-0}^{j} a_{n_c-m}^c a_{n-(j-m)} + K \sum_{m=0}^{j} b_{n_c-m}^c b_{n-(j-m)}; \quad j = 1,\ldots, n_p + n_c - 1$$

$$l_{n_p+n_c} = K + 1. \tag{17}$$

Corollary 4: The interval plant (1) is stabilizable by the controller (15) if and only if the corresponding four Kharitonov polynomials for the polynomial (16) with the parameters (17) are stabilizable by the selection of a_i^c and b_i^c .

The closed-loop characteristic polynomial can now be used to generate the Kharitonov polynomials. The stability of the interval plant using the n^{th} order controller (15) can be achieved by the stabilization of the four Kharitonov polynomials.

One possible approach for the stabilization of the four Kharitonov polynomials is the non-linear programming method suggested in (Arehart & Wolovich, 1995) for the stability of finite number of plants. In this approach, the stabilization of the polynomials is achieved by the solution of a feasibility non-linear programming problem. The constraints of this NLP problem are constructed using the Routh-Hurwitz criteria for the four Kharitonov polynomials.

Examples

Several examples are presented in this section to illustrate the proposed design method.

Example 1: **Dynamic of an Experimental Aircraft**
Consider the interval plant for the model of an experimental oblique aircraft [1]. The model is given by

$$G(s,c) = \frac{b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 a_0}$$

with parameters having uncertainties as given by

$b_0 \in [90, 166]$, $b_1 \in [54, 74]$, $a_3 \in [2.8, 4.6]$, $a_2 \in [50.4, 80.8]$, $a_1 \in [30.1, 33.9]$, $a_0 \in [-.1, .1]$

Consider the controller,

$$C(s) = K_p + \frac{K_I}{s}$$

Using proportional gain only, one may get the following four virtual systems

$$G_1^v = \frac{\bar{b}_1 s + \underline{b}_0}{s^4 + \underline{a}_3 s^3 + \bar{a}_2 s^2 + \bar{a}_1 s + \underline{a}_0}$$

$$G_2^v = \frac{\underline{b}_1 s + \underline{b}_0}{s^4 + \bar{a}_3 s^3 + \bar{a}_2 s^2 + \underline{a}_1 s + \underline{a}_0}$$

$$G_3^v = \frac{\bar{b}_1 s + \bar{b}_0}{s^4 + \underline{a}_3 s^3 + \underline{a}_2 s^2 + \bar{a}_1 s + \bar{a}_0}$$

$$G_4^v = \frac{\underline{b}_1 s + \bar{b}_0}{s^4 + \bar{a}_3 s^3 + \underline{a}_2 s^2 + \underline{a}_1 s + \bar{a}_0}$$

The study of the stability of these systems reveals that the stabilizing gain is given by $K_p \in [0.017, 1.248]$

Now consider a PI controller, which can be written in the form

$$C(s) = K\frac{s+z}{s}$$

with $K = K_p$, and $z = K_i/K_p$. For a given location of the zero of the PI controller, the problem is simplified to the problem of stabilizing using pure gain case. The four virtual systems are given by

$$G_1^v = \frac{\bar{b}_1 s^2 + (z\,\underline{b}_1 + \underline{b}_0)s + z\underline{b}_0}{s^5 + \underline{a}_3 s^4 + \bar{a}_2 s^3 + \bar{a}_1 s^2 + \underline{a}_0 s}$$

$$G_2^v = \frac{\bar{b}_1 s^2 + (z\,\bar{b}_1 + \bar{b}_0)s + z\underline{b}_0}{s^5 + \underline{a}_3 s^4 + \bar{a}_2 s^3 + \bar{a}_1 s^2 + \underline{a}_0 s}$$

$$G_3^v = \frac{\underline{b}_1 s^2 + (z\,\bar{b}_1 + \bar{b}_0)s + z\,\bar{b}_0}{s^5 + \bar{a}_3 s^4 + \underline{a}_2 s^3 + \underline{a}_1 s^2 + \bar{a}_0 s}$$

$$G_4^v = \frac{\underline{b}_1 s^2 + (z\,\underline{b}_1 + \underline{b}_0)s + z\,\bar{b}_0}{s^5 + \bar{a}_3 s^4 + \bar{a}_2 s^3 + \underline{a}_1 s^2 + \underline{a}_0 s}$$

For each of the above systems, a root locus may be generated. From these plots, one can get information about the range of stabilizing gain K_p for a given selection of z. The values for the controller parameters K_p and K_i that stabilize the system can now be found. The region we obtained is slightly different from the one reported in Barmish *et al.* (1992) may be due to computational errors in the development in Barmish *et al.* (1992). We again draw the attention of the reader to the simplicity and the less amount of computations needed in our design approach.

Example 2

Consider now the stabilization of the system

$$G(s) = \frac{b_0 + s}{a_0 + a_1 s + a_2 s^2 + a_3 s^3}$$

with the bounds on the uncertainties given by $b_0 \in [1.,7], a_0 \in [1,5], a_1 \in [3, 10], a_3 = 1$

The controller is a lead-lag given by

$$C(s) = K\frac{s+z}{s+p}$$

The root locus for the four virtual systems using pure gain controller suggested that the system cannot be stabilized by pure gain. However, locations for the pole and the zero of the controller may be selected so that the asymptotic behavior of the roots are forced in the stable region. The selection of the zero of the controller at s =-1.5, and the location of the pole at s =-15.0 makes the system stable for all K>13.2. For this example, virtual system number four is the critical one, and all other virtual systems are stable for all positive gain K.

Example 3

Consider the design of a second-order controller for the system given in example 1. The controller is given by

$$C(s) = K\frac{s^2 + b_1^c s + b_0^c}{s^2 + a_1^c s + a_0^c}$$

The closed-loop characteristic polynomial is given by

$$\Delta_{cl}(s) = l_0 + l_1 s + \dots\dots + l_6 s^6$$

with

$$\begin{aligned}
l_0 &= a_0 a_0^c + K b_0 b_0^c \\
l_1 &= a_0 a_1^c + a_1 a_0^c + K\left(b_1 b_0^c + b_0 b_1^c\right) \\
l_2 &= a_0 + a_1 a_1^c + a_2 a_0^c + K\left(b_0 + b_1 b_1^c + b_2 b_0^c\right) \\
l_3 &= a_1 + a_2 a_1^c + a_3 a_0^c + K\left(b_1 + b_2 b_1^c + b_3 b_0^c\right) \\
l_4 &= a_1 + a_3 a_1^c + a_0^c + K\left(b_2 + b_3 b_1^c + b_0^c\right) \\
l_5 &= a_3 + a_1^c + K\left(b_3 + b_1^c\right) \\
l_6 &= K + 1
\end{aligned}$$

and the four Kharitonov polynomials are given by

$$\begin{aligned}
\Delta_{cl}^1(s) &= \underline{l}_0 + \underline{l}_1\, s + \bar{l}_2\, s^2 \dots\dots\dots\dots \\
\Delta_{cl}^2(s) &= \underline{l}_0 + \bar{l}_1\, s + \bar{l}_2\, s^2 \dots\dots\dots\dots \\
\Delta_{cl}^3(s) &= \bar{l}_0 + \bar{l}_1\, s + \underline{l}_2\, s^2 \dots\dots\dots\dots \\
\Delta_{cl}^4(s) &= \bar{l}_0 + \underline{l}_1\, s + \underline{l}_2\, s^2 \dots\dots\dots\dots
\end{aligned}$$

with the coefficients of these polynomials found as functions of the upper and the lower bounds of the system's parameters.

The proposed NLP based approach was used to find the controller parameters which stabilizes the above four polynomials. The resulting controller is given by

$$C(s) = 0.265\frac{s^2 + 5.05s + 5.18}{s^2 + 2.27s + 1.69}$$

Conclusions

A simple low-order controller design procedure is proposed for interval plants. The proposed method makes use of the Kharitonov theorem and the root locus. Also, a non-linear programming based method is proposed for the design of higher order controllers. Examples illustrating the design schemes have been presented.

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