ORIGINAL PAPER

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# A Nine Point Formula to Approximate the Laplace Operator for Irregular Domains

**Abstract:** A nine point formula to approximate the Laplace operator for irregular domains has been derived using the Taylor series. This formula proved to be of order $O(h^2)$. A comparison with those methods on finite difference and finite element has shown that the above formula is more accurate.

**Keywords:** Numerical Analysis, Laplace operator, Finite Difference

* This work was completed when the author was studying for a Ph.D degree at Comenius University in Slovakia in 1993.

**تقريب مؤثر لا بلاس بتسعة نقاط في نطاق غير منتظم**

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المستخلص: اشتقاق مؤثر لا بلاس في نطاق غير منتظم بتسعة نقاط باستعمال متسلسلة تايلور. برهنت هذة الصيغة بأنها من ا لرتبة $O(h^2)$ بالمقارنة مع طرق الفروق المنتهية والعناصر المنتهية ثبت انها اكثر دقة.

كلمات مدخلية: تحليل عددي-مؤثر لابلاس-مقارنة-طرق الفروق المتنحية-العناصر المنتهية

## Introduction:

Let $H$ and $X$ be partial differential operators associated with the Laplace differential operator in two dimensions defined by

$$HF(x, y) = F(x+h, y) + F(x-h, y) + F(x, y+h) + F(x, y-h) - 4F(x,y)$$

$$XF(x, y) = (1/2)[F(x+h, y+h) + F(x-h, y+h) + F(x-h, y-h) + F(x+h, y-h) - 4F(x, y)]$$

each of them approximate $\nabla^2$ in a regular domain i.e.

$$h^2 \nabla^2 u = Hu + O(h^4) \qquad [6]$$

$$h^2 \nabla^2 u = Xu + O(h^4) \qquad [6]$$

Another operator depending on nine points given by

$$h^2 \nabla^2 u = Ku + O(h^4) \text{ where } K = 4H + 2X \qquad [3], [6]$$

is more accurate than both $H$ and $X$.

In irregular domains $H$ and $X$ were presented, [6], as follows:

$$(1) \qquad \overline{H} = \frac{2}{h_1 + h_3}\left[\frac{u(x+h_1, y) - u(x,y)}{h_1} + \frac{u(x-h_3, y) - u(x, y)}{h_3}\right] +$$

$$\frac{2}{h_2 + h_4}\left[\frac{u(x, y+h_2) - u(x, y)}{h_2} + \frac{u(x, y-h_4) - u(x, y)}{h_4}\right]$$

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$$(2)\qquad \overline{X} = \frac{1}{h_5 + h_7}\left[\frac{u(x + h_5, y + h_5) - u(x,y)}{h_5} + \frac{u(x - h_7, y + h_7) - u(x, y)}{h_7}\right] +$$

$$\frac{1}{h_6 + h_8}\left[\frac{u(x - h_6, y + h_6) - u(x, y)}{h_6} + \frac{u(x + h_8, y - h_8) - u(x, y)}{h_8}\right]$$

together with many methods that approximate points near the boundary for irregular domains, which, in applications, causes calculation difficulties.

In this work an operator depending on nine points, seven of which may be non-nodel points, has been produced. This operator is $K = 4H + 2X$, i. e.

$$(3)\qquad \overline{K}u = \nabla^2 u = \sum_{i=0}^{8} \alpha_i u_i, \text{ where}$$

$$(4)\qquad \alpha_1 = \frac{8}{h^2 s_1 (s_1 + s_3)},\ \alpha_2 = \frac{8}{h^2 s_2 (s_2 + s_4)},\ \alpha_3 = \frac{8}{h^2 s_3 (s_3 + s_1)}$$

$$\alpha_4 = \frac{8}{h^2 s_4 (s_4 + s_2)},\ \alpha_5 = \frac{2}{h^2 s_5 (s_5 + s_7)},\ \alpha_6 = \frac{2}{h^2 s_6 (s_6 + s_8)}$$

$$\alpha_7 = \frac{2}{h^2 s_7 (s_7 + s_5)},\ \alpha_8 = \frac{2}{h^2 s_8 (s_8 + s_6)},\ \text{and}$$

$\alpha = -\sum_{i=0}^{8} \alpha_i$ where $h$ is $h_i = s_i\, h$, $0 < s_i \le 1$, i=1,2,3,4,5,6,7,8, and $h$ is the step length of the grid.

$\overline{K}$ proved to be of order O $(h^2)$. By applying $\overline{K}$ to Laplace and Poisson equations in different irregular domains, it has been shown that the results are more accurate than those in other methods based on finite difference and finite element.

## The $\overline{K}$ operator

The operator $\overline{K}$ is derived as a linear combination of $\overline{H}$ and $\overline{X}$ for the domain with curved boundary and irregular nodal points near the boundary. The Taylor series was used to approximate eight points, $u(x+h_1,y)$, $u(x-h_3,y)$, $u(x,y+h_2)$, $u(x,y-h_4)$, $u(x+h_5,y+h_5)$, $u(x-h_6,y+h_6)$, $u(x-h_7,y-h_7)$, and $u(x+h_8,y-h_8)$ as follows:

Let $\alpha^i = \frac{\partial^i}{\partial \chi^i}$, $\beta = \frac{\partial^i}{\partial y^i}$, i =1,2,3,... . , then ;

(2.1) $u_1 = u\,(x + h_1,y) = u + h_1\alpha u + (h_1^2 / 2!)\alpha^2 u + (h_1^3 / 3!)\alpha^3 u + .....$

(2.2) $u_2 = u\,(x, y + h_2) = u + h_2\beta u + (h_2^2 / 2!)\beta^2 u + (h_2^3 / 3!)\beta^3 u + .....$

(2.3) $u_3 = u\,(x - h_3,y) = u - h_3\alpha u + (h_3^2 / 2!)\alpha^2 u + (h_3^3 / 3!)\alpha^3 \underline{u} + .....$

(2.4) $u_4 = u\,(x, y - h_4) = u - h_4\beta u + (h_4^2 / 2!)\beta^2 u + (h_4^3 / 3!)\beta^3 u + .....$

(2.5) $u_5 = u\,(x + h_5,y + h_5) = u + h_5(\alpha + \beta)u + (h_5^2 / 2!)(\alpha + \beta)^2 u + ....$

(2.6) $u_6 = u\,(x - h_6,y + h_6) = u + h_6(-\alpha + \beta)u + (h_6^2 / 2!)(-\alpha + \beta)^2 u + ....$

(2.7) $u_7 = u\,(x - h_7,y - h_7) = u + h_7(-\alpha - \beta)u + (h_7^2 / 2!)(-\alpha - \beta)^2 u + ....$

(2.8) $u_8 = u\,(x - h_8,y - h_8) = u + h_5(\alpha - \beta)u + (h_8^2 / 2!)(\alpha - \beta)^2 u + ....$

The Laplace operator is obtained from the equations (2.1) - (2.4) and from (2.5) - (2.8) and by multiplying the first by 4 and the second by 2, we can add the two results to get the following equation:

(2.9) $$\sum_{i=0}^{8} a_i u_i = 6(\alpha^2 + \beta^2)u + K., \text{ where}$$

(2.10) $$\alpha_1 = \frac{8}{h^2 s_1 (s_1 + s_3)},\ \alpha_2 = \frac{8}{h^2 s_2 (s_2 + s_4)},\ \alpha_3 = \frac{8}{h^2 s_3 (s_3 + s_1)},\ \alpha_4 = \frac{8}{h^2 s_4 (s_4 + s_2)},$$

$$\alpha_5 = \frac{2}{h^2 s_5 (s_5 + s_7)},\ \alpha_6 = \frac{2}{h^2 s_6 (s_6 + s_8)},\ \alpha_7 = \frac{2}{h^2 s_7 (s_7 + s_5)},\ \alpha_8 = \frac{2}{h^2 s_8 (s_8 + s_6)}, \text{ and}$$

(2.11) $\alpha_0 = -\sum_{i=1}^{8} \alpha_i$, where $h_i = s_i\, h$, $0 < s_i \leq 1$, i = 1,2,3,4,5,6,7,8, and $h$ is the step length of the grid.

Now we have

(2.12) $$L_h[u] = K\, u = \sum_{i=0}^{8} \alpha_i\, u_i = 6(\alpha^2 + \beta^2)u + \ldots.$$

**Solvability of the difference equation by $\overline{K}$**

Consider the linear second order partial differential equation of the form

(3.1) $$L_h[u] = u_{xx} + u_{yy} + G$$

In the application of the finite difference [9] one replaces the region $R$ by a set of points $R_h$, where $R_h \subseteq R$ and also replaces the boundary $S$ by $S_h$. $R_h^*$ represents the set of all regular points, and $R_h'$ represents the irregular points. The application of $\overline{K}$ to the Laplace or Poisson equations with boundary conditions, leads to a system of linear algebraic equations of the form

(3.2) $$Au = b,$$

where A is a $n \times n$ matrix, b is a known $n \times 1$ matrix, and $u$ is a solution $n \times 1$ matrix. Here $n$ is the number of interior points in the grid. Thus, it follows from (2.10) that in the difference equation (2.9) we have

(3.3) $$-\alpha_0 = -\sum_{i=1}^{8} \alpha_i,\ \alpha_i > 0, \quad i = 1,2,3,4,5,6,7,8.$$

The uniqueness of the solution can be easily proved (see [9]).

**Accuracy of the difference equation in $\overline{K}$**

We now investigate the accuracy of the solution of difference equation. Our analysis is [4], and we assume that the exact solution of the differential equation has partial derivatives of all orders up to and including the fourth which are bounded in $R + S$. We need the following Lemmas, which are useful later.

**Lemma 1**: Let $L_h[u]$ be a discrete operator of the form

(4.1) $L_h[u] = \sum_{i=0}^{8} \alpha_i\, u_i = G$, where $-\alpha_0, \alpha_1, \ldots, \alpha_8$, are positive functions such that

(4.2) $\quad -\alpha_0 \geq \alpha_1 + \ldots + \alpha_8.$

if $u \geq 0$ on $S_h$ and $-L_h[u] \geq 0$ on $R_h$, then $u \geq 0$ in $R_h$.

**Proof:**

If $u > 0$ for some point of $R_h$, then for some point $(\bar{x}, \bar{y})$ of $R_h$, we have $u(\bar{x}, \bar{y}) \leq u(x,y)$ for all $(x,y) \in R_h$ and $u(\bar{x}, \bar{y})<0$.

Let $M = -u(\bar{x}, \bar{y})$. We seek to show that $u(\bar{x}+h, \bar{y}), u(\bar{x}, \bar{y}+h), \ldots$, are equal to $-M$. But since $L_h[u] \quad u(\bar{x}, \bar{y}) \leq 0$, we have

(4.3) $$-\alpha_0\, u(\bar{x}, \bar{y}) \geq \sum_{i=I}^{8} \alpha_i \bar{u}_i$$

therefore

(4.4) $$0 \geq \sum_{i=I}^{8} \alpha_i \bar{u}_i = \sum_{i=I}^{8} \alpha_i [\bar{u}_i + M]$$

since $\alpha_i > 0$, $\bar{u}_i + M \geq 0$, i = 1,2,...,8. The last expression can be nonpositive only if $h_i = -M$, i = 1,2,...,8. In a similar way, we can show that $u(\bar{x} + 2h, \bar{y})$, $u(\bar{x}, \bar{y} - 2h)$, $u(\bar{x} - 2h, \bar{y})$, $u(\bar{x}, \bar{y} - 2h)$, $u(\bar{x} + 2h, \bar{y} + 2h)$, $u(\bar{x} - 2h, \bar{y} + 2h)$, $u(\bar{x} - 2h, \bar{y} - 2h)$, and $u(\bar{x} + 2h, \bar{y} - 2h)$ are equal to $-M$ for all $(x, y)$ in $R_h+S_h$. But since $u \geq 0$ in $S_h$, we have a contradiction. Hence $u \geq 0$ in $R_h + S_7$.

**Lemma 2:** Let $\bar{u}$ satisfy $L_h[u] = G$, and satisfy the boundary condition $\bar{u} = g$. Let $u$ satisfy $L_h[u] = G$ in $R_h$, if $\bar{u}$ has partial derivatives of all orders up to the fourth order which are continuous and bounded in $R + S$, then

(4.5) $$\left| L_h[u - \bar{u}] \right| \leq h^2 \frac{M_4}{6}, \text{ where } M_4 = \max \left\{ \max_{R+S} \left| \frac{\partial^4 u}{\partial x^4} \right|, \max_{R+S} \left| \frac{\partial^4 u}{\partial 4 y^4} \right| \right\}$$

(For the proofs see [9] ch. (10) and (15)).

**Theorem :** Under the hypotheses of lemma 2, for all $(x,y) \in R_h+S_h$ we have

(4.6) $$\left| u(x,y) - \bar{u}(x,y) \right| \leq \frac{h^2 r^2}{24}, \max_{R+S} [M_4] + \max_{S_h} \left| u(x,y) - \bar{u}(x,y) \right|$$

where $r$ is the radius of the circle which contains $R + S$.

**Proof:** The first term on the right side was already computed in [9]. To the second term, we use in $R_h$, the equation:

(4.7) $$L_h[u] = \sum_{i=0}^{8} \alpha_i u_i = G.$$

Solving for $u(x,y)$ we have

(4.8) $$u(x,y) = \frac{8 s_2 s_4 s_5 s_6 s_7 s_8}{Q} \left\{ \frac{s_3}{s_1 = s_3} u_1 + \frac{s}{s_1 + s_3} u_3 \right\} + \frac{8 s_1 s_3 s_5 s_6 s_7 s_8}{Q} \left\{ \frac{s_4}{s_2 + s_4} u_2 + \frac{s_2}{s_2 + s_4} u_4 \right\} +$$

$$\frac{2s_1s_2s_3s_4s_6s_8}{Q}\left\{\frac{s_7}{s_7 = s_5}u_5 + \frac{s_5}{s_5+s_7}u_7\right\} + \frac{2s_1s_2s_3s_4s_5s_7}{Q}\left\{\frac{s_8}{s_6+s_8}u_2 + \frac{s_6}{s_6+s_8}u_8\right\},$$

where, $Q = 8s_2s_4s_5s_7s_6s_8 + 8s_1s_3s_5s_7s_6s_8 + 2s_1s_2s_3s_4s_5s_8 + 2s_1s_2s_3s_4s_5s_6$.

The first expression in the brackets corresponds to the linear interpolation in the points $u(x+h_1,y)$ and $u(x-h_3,y)$, the second corresponds to linear interpolation in the points $u(x,y+h_2)$ and $u(x,y-h_4)$. The third is the linear interpolation in the points $u(x+h_5,y+h_5)$ and $u(x-h_7,y-h_7)$, and the fourth is the linear interpolation in the points $u(x-h_6,y+h_6)$ and $u(x+h_8,y-h_8)$. The overall expression represents linear interpolations in two interpolated values. By the properties of linear interpolation we have:

(4.9)
$$\left|\bar{u}(x,y) - \left[\frac{s_3}{s_1+s_3}\bar{u}_1 + \frac{s_1}{s_1+s_3}\bar{u}_3\right]\right| \le \frac{h^2(s_1+s_3)^2}{8}M_2$$

$$\left|\bar{u}(x,y) - \left[\frac{s_4}{s_2+s_4}\bar{u}_2 + \frac{s_2}{s_2+s_4}\bar{u}_4\right]\right| \le \frac{h^2(s_2+s_4)^2}{8}M_2$$

$$\left|\bar{u}(x,y) - \left[\frac{s_7}{s_5+s_7}\bar{u}_2 + \frac{s_5}{s_5+s_7}\bar{u}_7\right]\right| \le \frac{h^2(s_5+s_7)^2}{8}M_2$$

$$\left|\bar{u}(x,y) - \left[\frac{s_8}{s_6+s_8}\bar{u}_2 + \frac{s_6}{s_6+s_8}\bar{u}_8\right]\right| \le \frac{h^2(s_6+s_8)^2}{8}M_2$$

where $u_1 = u(x+h_1,y)$, $u_2 = u(x,y-h_2)$, $u_3 = u(x-h_3,y)$, $u_4 = u(x,y-h_4)$, $u_5 = u(x+h_5, y+h_5)$, $u_6 = u(x-h_6, y+h_6)$, $u_7 = u(x-h_7, y-h_7)$, $u_8 = u(x+h_8, y-h_8)$ and

$$M_2 = \max\left|\frac{\partial^2 u}{\partial x^2}\right|, \left|\frac{\partial^2 u}{\partial y^2}\right|$$

(4.10)
$$\bar{u}(x,y) - \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left\{\frac{s_3}{s_1+s_3}\bar{u}_1 + \frac{s_1}{s_1+s_3}\bar{u}_3\right\}$$

$$- \frac{8s_1s_2s_5s_6s_7s_8}{Q}\left\{\frac{s_4}{s_2+s_4}\bar{u}_2 + \frac{s_2}{s_2+s_4}\bar{u}_4\right\}$$

$$- \frac{2s_1s_2s_3s_4s_6s_8}{Q}\left\{\frac{s_7}{s_5+s_7}\bar{u}_5 + \frac{s_5}{s_5+s_7}\bar{u}_7\right\}$$

$$- \frac{2s_1s_2s_5s_4s_5s_8}{Q}\left\{\frac{s_8}{s_6+s_8}\bar{u}_6 + \frac{s_6}{s_6+s_8}\bar{u}_8\right\}\Bigg| \le \frac{h^2}{2}M_2$$

since $S_i \le 1$, i = 1, 2,...,8. Thus we have

(4.11)
$$|u(x,y) - \bar{u}(x,y)| \le \gamma_1|e(x+h_1, y)| + \gamma_2|e(x, y+h_2)|$$

$$+ \gamma_3 e(x-h_3, y)| + \gamma_4 |e(x, y-h_4)|$$

$$+ \gamma_5|e(x+h_5, y+h_5)| + \gamma_6|e(x-h_6, y+h_6)|$$

$$+ \gamma_7|e(x-h_7, y-h_7)| + \gamma_8|e(x+h_8, y-h_8)| + \frac{h^2}{2}M_2$$

where $e(x, y) = u(x, y) - \bar{u}(x, y)$, and

$$(4.12) \qquad \gamma_1 = \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left\{\frac{s_3}{s_1+s_3}\right\} \quad \gamma_3 = \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left\{\frac{s_1}{s_1+s_3}\right\}$$

$$\gamma_2 = \frac{8s_1s_3s_5s_6s_7s_8}{Q}\left\{\frac{s_4}{s_2+s_4}\right\} \quad \gamma_4 = \frac{8s_1s_3s_5s_6s_7s_8}{Q}\left\{\frac{s_2}{s_2+s_4}\right\}$$

$$\gamma_5 = \frac{2s_1s_2s_3s_4s_6s_8}{Q}\left\{\frac{s_7}{s_5+s_7}\right\} \quad \gamma_7 = \frac{2s_1s_2s_3s_4s_6s_8}{Q}\left\{\frac{s_5}{s_5+s_7}\right\}$$

$$\gamma_6 = \frac{2s_1s_2s_3s_4s_5s_7}{Q}\left\{\frac{s_8}{s_6+s_8}\right\} \quad \gamma_8 = \frac{2s_1s_2s_3s_4s_5s_7}{Q}\left\{\frac{s_6}{s_6+s_8}\right\}$$

We now seek to show that for $(x,y) \in R_h'$.

$$(4.13) \qquad \left| u(x, y) - \bar{u}(x, y) \right| \leq \frac{4}{5} \max_{R_h} e(x, y,) + \frac{h^2}{2} M_2$$

To do this, first consider the case that one of the points, say $(x+h_1,y)$ is in $S$ and the other points are not. Hence

$s_2 = s_3 = ... = s_8 = 1$, in this case $e(x + h_1, y) = 0$ since $\sum_{i=0}^{8}$, $\gamma_1 = 1$, and since

$$(4.14) \qquad \gamma_2 + \gamma_4 = \frac{8s_1s_3s_5s_6s_7s_8}{Q} \leq = \frac{8s_2s_4s_5s_6s_7s_8}{Q} = \gamma_1 + \gamma_3$$

we have $\gamma_1 + \gamma_3 \geq \frac{2}{5}$. Moreover, since

$$(4.15) \qquad \frac{s_3}{s_1+s_3} \geq \frac{s_1}{s_1+s_3}, \text{ we have } \gamma_1 \geq \gamma_3 \text{ and } \gamma_1 \geq \frac{1}{5}.$$

Thus $\sum_{i=2}^{8} \gamma l \leq \frac{4}{5}$, then (14) holds.

Now consider the case where two of the points are on $S$. There are essentially two different cases. In the first case $(x+h_1,y)$ and $(x-h_3, y)$ are in $S$ and the other six points are not. We have

$\gamma_1 + \gamma_3 \geq \gamma_2 + \gamma_4$. Since $\sum_{i=1}^{8} = 1$, and $\gamma_1 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 \leq \frac{2}{5}$. Thus we have

$$(4.16) \qquad \left| u(x, y) - \bar{u}(x, y) \right| \leq \frac{2}{5} \max_{R_h} \left| e(x, y,) \right| + \frac{h^2}{2} M_2.$$

A similar arrangement holds if $(x,y+h_2)$ and $(x,y-h_4)$ are in S, or $(x+h_5,y+h_5)$ and $(x-h_7,y-h_7)$ are in $S$, or $(x-h_6,y+h_6)$ and $(x+h_8,y-h_8)$ are in $S$. If $(x+h_1,y)$ and $(x-h_3,y)$ are in $S$ we have $\gamma_1 \geq \gamma_3$, $\gamma_2 \geq \gamma_4$ and $\gamma_1 + \gamma_2 \geq \gamma_3 + \gamma_4$ *so that* $\gamma_1 + \gamma_2 \geq \frac{2}{5}$ *and*

$$(4.17) \qquad \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8 \leq \frac{3}{5} \max_{R_h} \left| e(x, y) \right| + \frac{h^2}{2} M_2.$$

Similarly in the other cases it holds if any two different points lie on $S$.
Next, let us consider the case where only one of the points, say $(x+h_1,y)$ is not in $S$. Evidently

(4.18) $$\gamma_2 = \frac{8s_2s_4s_5s_6s_7s_8}{Q}\left\{\frac{s_3}{s_1+s_3}\right\} \leq \frac{2}{5}\left\{\frac{s_3}{s_1+s_3}\right\} \leq \frac{1}{5}$$

since $8s_2s_4s_5s_6s_7s_8\,(Q)^{-1}$ is an increasing function of $s_2s_4s_5s_6s_7s_8$ and $s_2s_4s_5s_6s_7s_8 \leq 1$, and since $s_3(1+s_3)^{-1}$ is an increasing function of $s_3$ and $s_3 \leq 1$. Thus we have

(4.19) $$\left|u(x,y) - \bar{u}(x,y)\right| \leq \frac{1}{5}\max_{R_h}\left|e(x,y,)\right| + \frac{h^2}{2}M_2.$$

A similar discussion is used for when three or more, up to seven points do not lie on $S$. Finally, if all eight points $(x+h_1,y)$,..., etc are on $S$ we have

$$\left|u(x,y) - \bar{u}(x,y)\right| \leq \frac{h^2}{5}M_2$$

Therefore (4.13) holds in all cases.

We now let

(4.20) $$v = \max_{R_h}\left|e(x,y,)\right|, \mu = \max_{R_h}\left|e(x,y,)\right|$$

Evidently, from (4.14)

(4.21) $$v \leq \frac{h^2r^2M_4}{24} + \mu,\ \ m \leq \frac{4}{5}\max\{v,\mu\} + \frac{h^2}{2}M_2$$

If $v \leq \mu$, then $\mu \leq \frac{4}{5}v + \frac{h^2}{2}M_2$, and

(4.22) $$v \leq \frac{h^2r^2}{24}M_4 + \frac{4}{5}v + \frac{h^2}{2}M_2$$

or

(4.23) $$v \leq \frac{5h^2r^2}{24}M_4 + \frac{5}{2}h^2M_2.$$

On the other hand, if $\mu \geq v$, then

(4.24) $$\mu \leq \frac{4}{5}\mu + \frac{h^2}{2}M_2,\ and$$

(4.25) $$\mu \leq \frac{4}{5}h^2M_2,$$

therefore since $\mu \leq v$ or else $\mu \leq \frac{5}{4}h^2M_2$, we have,

(4.26) $$\max_{R_h}\left|u(x,y) - \bar{u}(x,y)\right| \leq \frac{5h^2r^2}{24}M_4 + \frac{5}{2}h^2M_2$$

## Implementation:

In the following, different methods have been implemented for different examples that have curved boundaries and the results are presented in tables.

1- Laplace equation $\nabla^2 u = 0$ in the first quadrant bounded by the circle $x^2 + y^2 = 1$, where the exact solution and the boundary conditions are given by the equation $u(x, y) = \exp(-2x)\cos(2y)$.

| Step length | Max. absolute error in $\bar{H}$ | Max absolute error in $\bar{K}$ |
|---|---|---|
| 0.25 | 2.3409123E-03 | 8.961037E-04 |
| 0.125 | 6.51036E-04 | 6.16542E-05 |
| 0.0625 | 1.34793E-04 | 6.107614E-06 |
| 0.03125 | 4.41378958E-05 | 1.8039060E-06 |
| 0.015625 | 1.01590804E-05 | 2.234243580E-07 |

2- $\nabla^2 u = f$ in the first quadrant bounded by $x^2+y^2 = 1$, where the exact solution and the boundary conditions are given by $u(x,y) = \sin(\pi x)\sin(\pi y)$.

| Step length $h$ | Max. absolute error in $\bar{H}$ | Max. absolute error in $\bar{K}$ |
|---|---|---|
| 0.5 | 1.039023E-01 | 3.2436646E-02 |
| 0.25 | 3.135872E-02 | 2.3620913E-02 |
| 0.125 | 8.944179E-03 | 3.7283951E-03 |
| 0.0625 | 2.384006E-03 | 5.4587562E-04 |
| 0.03125 | 6.097213E-04 | 7.0241315E-05 |
| 0.015625 | 1.511102E-04 | 9.7525401E-06 |

3- Quarter moon with the exact solution and boundary conditions given by, $u(x,y)=\sin(\pi x)\sin(\pi y)$.

| Method of Solutions | Max. absolute error |
|---|---|
| $\bar{H}$ | 1.20457E-03 |
| $\bar{K}$ | 2.787202E-04 |
| Finite element method (PLTMG package) | 7.54E-03 |

## Conclusion

In this work a finite difference operator $\bar{K}$ has been derived. This operator has been implemented in different cases together with the method based on finite difference and finite element and shown to be more accurate. Also in regular domains, $\bar{K}$ will be reduced to K. $\bar{K}$ may be used to compute the eigen values for irregular domains, in particular the wave guides for eccentric circles.

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