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High Accuracy Piecewise Approximation for Planar Curves

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ABSTRACT. A cubic piecewise approximation method is described for planar curves. The order of classical piecewise approximations is improved. The method exploits the freedom in the choice of the parametrization and raises the approximation order to 6. The cubic approximant and the curve have contact of second order. The examples show the simplicity of the construction and the Figures show the efficiency and the qualitative results of this approximation method.

Parametric polynomials are used in Computer Aided Geometric Design for approximation and interpolation purposes or, more generally, for geometric modelling applications. The question of how to approximate curves within a certain tolerance by polynomials and splines arises often in working with CAGD; various error estimations have been obtained (Boehm *et al.* 1984, Farin 1988, Hoschek and Lasser 1989, Yamaguchi 1988). In this paper we write down a cubic piecewise approximation procedure for planar curves which significantly raises the standard piecewise approximation rate to order 6. The cubic approximant has a second order contact (*i.e.* curvature continuity) at each node of the segment. This improvement for the cubic piecewise approximant was obtained first by de Boor *et al.* (1988) by generalization of cubic Hermite interpolation. In addition to position and tangent, the curvature is also prescribed at both end points of the segment. This method yields G^2

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parametric cubics with 6^{th} order accuracy. In (Rababah 1992, 1993, 1995) we have described approximation methods for planar curves which improve the standard rate obtained by local Taylor approximation and achieve the order [4m/3], where m is the degree of the approximating polynomial. The best approximation order 2m is achieved for a class of curves of nonzero measure. The methods are not only quantitive improvements but also qualitative improvements over the Taylor expansion. Degen (1993) also constructed a cubic rational approximant, which approximates with order 8. There are more related results for special cases in (Dannenberg and Nowaki 1985, Dokken *et al.* 1990, Goodman and Unsworth 1988, Hanna *et al.* 1986, Höllig 1988, Klass 1983, Sakai and Usmani 1990, Sederberg and Kakimoto 1990).

Mathematical Description

Let $C : t \rightarrow (f(t), g(t)),\ t \in I := [0, h]$ be a regular smooth planar curve (*i.e.*, $(f'(t), g'(t)) \neq (0, 0),\ \forall t \in I$) of a certain differentiability class $C^6(I)$. We want to approximate C by a spline P. Normally, we divide the curve C into pieces C_i, $i = 1, ..., n$, where each C_i is the image of a subinterval $[t_{i-1}, t_i]$ of the whole interval I, and $t_i \in I$, $i = 0, ..., n$, are called the nodes of interpolation. The pieces C_i, $i = 1, ..., n$, are approximated by polynomial curves P_i, $i = 1, ..., n$, satisfying certain geometric conditions at both end points. Finally, we join these curves in $P = \cup_{i=1,...n} P_i$ to get a geometric spline. To simplify the notation, we write simply C for C_i and P for P_i. That is, we want to approximate C by a polynomial curve

$$P : t \rightarrow \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, t \in I$$

where X(t) and Y(t) are polynomials of degree 3.

Definition (Order of approximation)

The polynomial curve P approximates the curve C in the interval I with order k, if k is the biggest integer with the property:

$$\{f(t) - X(t)\} = O(t^k), \quad \{g(t) - Y(t)\} = O(t^k), \ \forall t \in I.$$

We choose here $X(t) = \sum_{i=0}^{3} a_i t^i$ and $Y(t) = \sum_{i=0}^{3} b_i t^i$. So the j^{th} derivative of X(t) and Y(t) at t = h are given by the derivatives of X(t) and Y(t) at t = 0 as follows, for j = 1, 2, 3:

$$X^{(j)}(h) = \sum_{i=j}^{3} \frac{X^{(i)}(0)}{(i-j)!} h^{i-j}$$

$$Y^{(j)}(h) = \sum_{i=j}^{3} \frac{Y^{(i)}(0)}{(i-j)!} h^{i-j},$$

where $X^{(j)}(t)$ and $Y^{(j)}(t)$ are the j^{th} derivatives of $X(t)$ and $Y(t)$. If we choose for $X(t)$ and $Y(t)$ the approximation polynomial of degree 3, then P approximates C in $[0, h]$ with order 4; *i.e.*,

$$\{f(t) - X(t)\} = O(h^4), \quad \{g(t) - Y(t)\} = O(h^4); \ t \in I.$$

Without loss of generality we may assume that

$$(f(0), g(0)) := (0, 0), \quad (f'(0), g'(0)) := (1,0),$$

so that for $t \in [0, h]$ we can parametrize C in the form

$$C : t \rightarrow X(t) \rightarrow \begin{pmatrix} X(t) \\ \phi(X(t)) \end{pmatrix}, t \in I.$$

Construction of the cubic approximant

Thus, P approximates C in $[0, h]$ with order 6 iff P and C have second order of contact at each end point, *i.e* iff

$$\left(\frac{d}{dt}\right)^j \{\phi(X(t)) - Y(t)\}|_{t=0} = 0, \ j = 1, 2$$

$$\left(\frac{d}{dt}\right)^j \{\phi(X(t)) - Y(t)\}|_{t=h} = 0, \ j = 0, 1, 2, \tag{1}$$

$$X(h) = h, \quad X(0) = Y(0) = 0.$$

That is, we need to solve the following nonlinear system of equations:

$$\emptyset^{(1)}(0)X^{(1)}(0) - Y^{(1)}(0) = 0,$$

$$\emptyset^{(2)}(0)X^{(1)2}(0) + \emptyset^{(1)}(0)X^{(2)}(0) - Y^{(2)}(0) = 0,$$

$$\emptyset(h) - Y(h) = 0,$$

$$\emptyset^{(1)}(h)X^{(1)}(h) - Y^{(1)}(h) = 0,$$

$$\emptyset^{(2)}(h)X^{(1)2}(h) + \emptyset^{(1)}(h)X^{(2)}(h) - Y^{(2)}(h) = 0,$$

$$X(h) = h,$$

where $\emptyset^{(i)}$, $X^{(i)}$ and $Y^{(i)}$ are the i^{th} derivatives of $\emptyset$, X and Y at $t = 0$ or $t = h$ as indicated.

The above system could be solved as follows: The first 2 equations are solved for $Y^{(1)}(0)$ and $Y^{(2)}(0)$ and then substituted into the 3rd equation for $Y(h)$. The resulting equation is solved for $Y^{(3)}(0)$. Substituting $Y^{(1)}(0)$, $Y^{(2)}(0)$ and $Y^{(3)}(0)$ into the 4th and 5th equations for $Y^{(1)}(h)$ and $Y^{(2)}(h)$, and also substituting $X^{(1)}(0)$, $X^{(2)}(0)$ and $X^{(3)}(0)$ for $X^{(1)}(h)$ and $X^{(2)}(h)$. We solve then the 4th equation for $X^{(3)}(0)$. We substitute the resulting value of $X^{(3)}(0)$ into the 5th equation and solve it for $X^{(2)}(0)$ as follows:

$$Y^{(1)}(0) = \emptyset^{(1)}(0)\, X^{(1)}(0),$$

$$Y^{(2)}(0) = \emptyset^{(2)}(0)\, X^{(1)2}(0) + \emptyset^{(1)}(0)\, X^{(2)}(0).$$

Expanding $Y(h)$ in the 3rd equation, we get

$$\emptyset(h) - Y(0) - Y^{(1)}(0)h - \frac{Y^{(2)}(0)}{2}\, h^2 - \frac{Y^{(3)}(0)}{6}\, h^3 = 0.$$

Since $Y(0) = 0$ and substituting the values of $Y^{(1)}(0)$ and $Y^{(2)}(0)$ from the 1st and 2nd equations, we get

$$Y^{(3)}(0) = \frac{6}{h^3}\left(\emptyset(h) - h\emptyset^{(1)}(0)\, X^{(1)}(0) - h^2\, \frac{\emptyset^{(2)}(0)\, X^{(1)2}(0) + \emptyset^{(1)}(0)\, X^{(2)}(0)}{2}\right).$$

From the 6th equation we have

$$X^{(3)}(0) = \frac{6}{h^3}\left(h - X^{(1)}(0)h - \frac{X^{(2)}(0)}{2}h^2\right).$$

Substituting these and the derivatives $X^{(i)}(0)$ and $Y^{(i)}(0)$ for $X^{(i)}(h)$ and $Y^{(i)}(h)$ into the 4th equation yields

$$\begin{gathered}\varnothing^{(1)}(h)\left(X^{(1)}(0) + X^{(2)}(0)h + \frac{3}{h}\left(h - X^{(1)}(0)h - \frac{X^{(2)}(0)}{2}h^2\right)\right)\\ -\left(\varnothing^{(1)}(0)\,X^{(1)}(0) + \varnothing^{(2)}(0)\,X^{(1)2}(0)h + \varnothing^{(1)}(0)\,X^{(2)}(0)h\right)\\ -\left(\frac{3}{h}\left(\varnothing(h) - \varnothing^{(1)}(0)X^{(1)}(0)h - \left(\frac{\varnothing^{(2)}(0)X^{(1)2}(0) + \varnothing^{(1)}(0)X^{(2)}(0)}{2}\right)h^2\right)\right) = 0\end{gathered}$$

After simplification, we get

$$\begin{gathered}6\{\varnothing^{(1)}(h)\,h - \varnothing(h)\} + 4h\,\{\varnothing^{(1)}(0) - \varnothing^{(1)}(h)\}\,X^{(1)}(0) + h^2\varnothing^{(2)}(0)\,X^{(1)2}(0) +\\ h^2\,\{\varnothing^{(1)}(0) - \varnothing^{(1)}(h)\}\,X^{(2)}(0) = 0.\end{gathered}$$

By setting

$$a := \varnothing^{(1)}(h) - \varnothing^{(1)}(0)$$

and excluding the case $\varnothing^{(1)}(h) = \varnothing^{(1)}(0)$ by assuming $a \neq 0$, we can solve for $X^{(2)}(0)$ as follows

$$X^{(2)}(0) = \frac{6\varnothing^{(1)}(h)h - 6\varnothing(h)}{ah^2} + \frac{4\varnothing^{(1)}(0) - 4\varnothing^{(1)}(h)}{ah}X^{(1)}(0) + \frac{\varnothing^{(2)}(0)}{a}X^{(1)2}(0).$$

Substituting into the 5th equation and rearranging yield

$$\left(\frac{-3\varnothing^{(2)}(h)b^2}{h^2a^3} + \frac{2c}{h^2}\right) + \frac{2a}{h}X^{(1)}(0) + \frac{3\varnothing^{(2)}(h)\varnothing^{(2)}(0)b}{a^2}X^{(1)2}(0) + \frac{\varnothing^{(2)}(h)h^2\varnothing^{(2)2}(0)}{4a^2}X^{(1)4}(0) = 0,$$

where

$$b := h\varnothing^{(1)}(0) - \varnothing(h)$$

$$c := \varnothing(h) - \varnothing^{(1)}(h)h$$

we solve this equation for $X^{(1)}(0)$ and using back substitution, we find the other derivatives of X and Y:

$$X^{(2)}(0) = \frac{-6c}{ah^2} - \frac{4}{h} X^{(1)}(0) + \frac{\phi^{(2)}(0)}{a} X^{(1)2}(0),$$

$$X^{(3)}(0) = \frac{-6(b-2c)}{ah^3} + \frac{6}{h^2} X^{(1)}(0) - \frac{3\phi^{(2)}(0)}{ah} X^{(1)2}(0),$$

$$Y^{(1)}(0) = \phi^{(1)}(0)\, X^{(1)}(0),$$

$$Y^{(2)}(0) = \frac{-6c\phi^{(1)}(0)}{ah^2} - \frac{4\phi^{(1)}(0)}{h} X^{(1)}(0) + \frac{\phi^{(1)}(h)\phi^{(2)}(0)}{a} X^{(1)2}(0),$$

$$Y^{(3)}(0) = \frac{6(2\phi^{(1)}(0)c - \phi^{(1)}(h)b)}{ah^3} + \frac{6\phi^{(1)}(0)}{h^2} X^{(1)}(0) - \frac{3\phi^{(1)}(h)\phi^{(2)}(0)}{ah} X^{(1)2}(0).$$

Examples and Figures

We have got a solution which coincides with the graph of the approximated function to all of the examples which we have studied. We write down some examples with figures showing the efficiency of the approximation method described in this paper. The software MATLAB have been used to calcualte and visualize the results. In the following figures, the high order cubic piecewise approximation (solid line ——) interpolates the given curve (dotted line ....) piecewise at the points, in the order (t, x, y).

Figure 1 shows the cubic piecewise approximation of the four-leaved rose

$$(2 \sin 2t \cos t,\ 2 \sin 2t \sin t).$$

interpolated piecewise at the points marked with the small circles(in the order t, x, y);

$$(\frac{\pi}{8}, 1.30656, 0.54119), \quad (\frac{3\pi}{8}, 0.54119, 1.30656),$$

$$(\frac{9\pi}{16}, 0.14931, -0.75066),\ (\frac{3\pi}{4}, 1.41421, -1.41421),$$

$$(\frac{99\pi}{100}, 0.12552, -0.00394),\ (\frac{5\pi}{4}, -1.41421, -1.41421),$$

$$(\frac{11\pi}{8}, -0.5412, -1.3066),\ (\frac{119\pi}{80}, -0.00616, -0.1568),$$

$$(\frac{13\pi}{8}, -0.5412, 1.3066),\ (\frac{7\pi}{4}, -1.41421, 1.41421),$$

and

$$(\frac{159\pi}{80}, -0.1568, 0.00616).$$

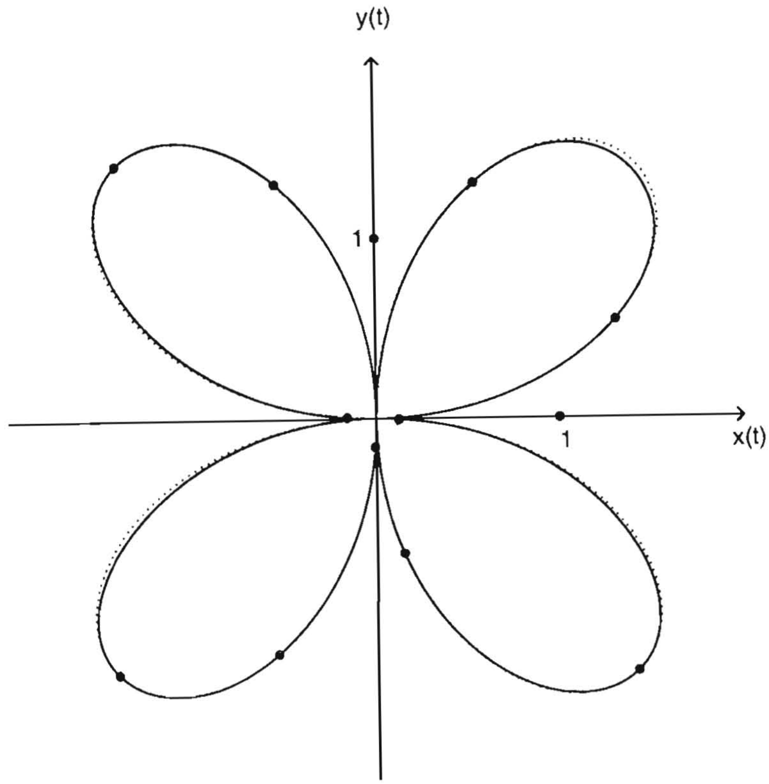

Fig. 1. The rose
• : Points of interpolation.

Figure 2 shows the cubic interpolant of the circle approximated at the points:

($\pi/8$, .92388, .38268), ($5\pi/8$, – 0.38268, .92388)

($9\pi/8$, – 0.92388, – 0.38268), ($13\pi/8$, –.38268, 0.92388)

Figure 3 shows the associated curvatures.

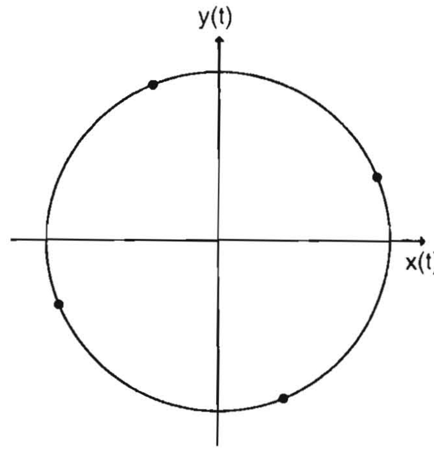

Fig. 2. The circle
• : Points of interpolation.

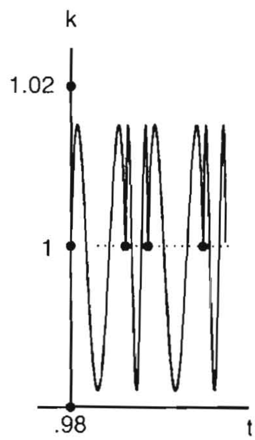

Fig. 3. The curvatures.

As shown in the error table, the error decays at the predicted rate $O(n^{-6})$, as the number n of interpolation points is increased.

Figure 4 shows the errors of interpolation (for only one segment, from left to right) of the circle at 4, 8, 16, 32 and 64 points multiplied by 10^2, 10^4, 10^6, 10^8 and 10^{10} respectively.

Error table

No. of points	error	rate
4	.195E-2	
8	.290E-4	– 6.07
16	.440E-6	– 6.04
32	.675E-8	– 6.02
64	1.08E-10	– 6.00

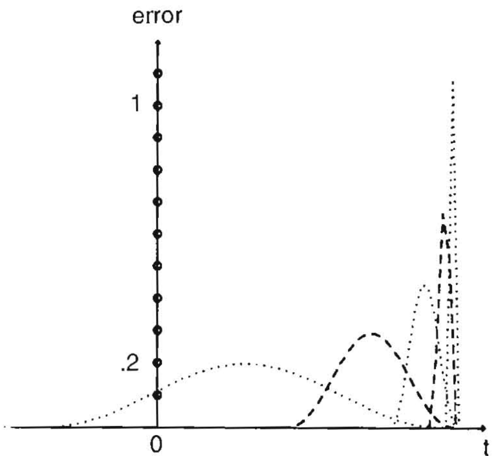

Fig. 4. The errors.

Figure 5 shows the hypotrochoid

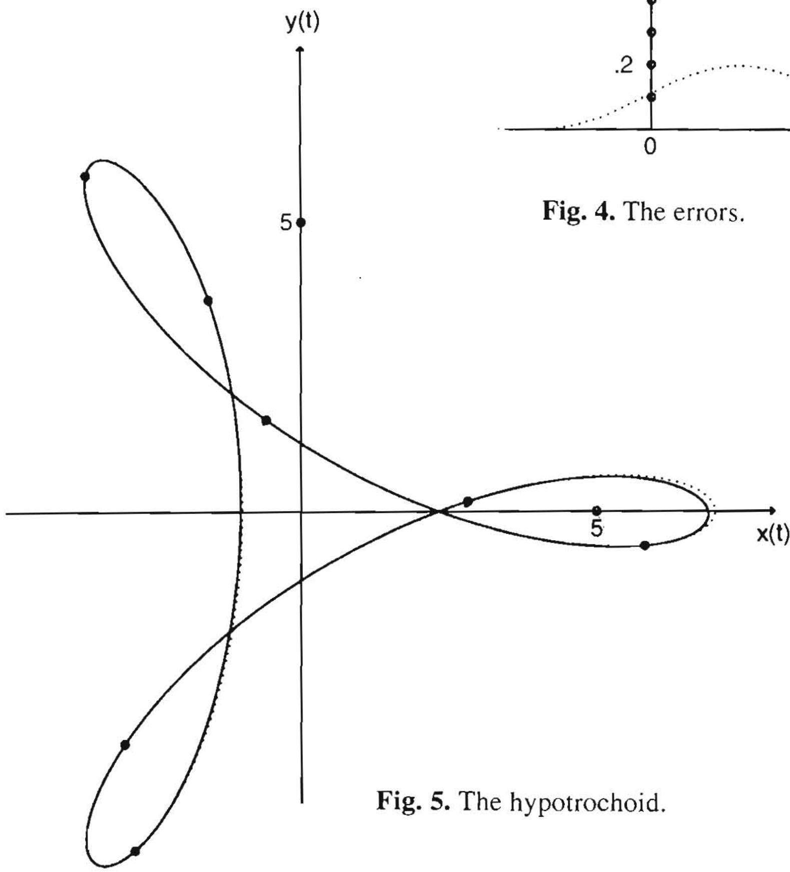

Fig. 5. The hypotrochoid.

$$(4 \cos t + 3 \cos 2t, 4 \sin t - 3 \sin 2t)$$

interpolated at the points

$(\pi/8, 5.8168, -0.5906)$, $(3\pi/8, -0.5906, 1.5742)$, $(5\pi/8, -3.6521, 5.8168)$

$(7\pi/8, -1.5742, 3.6521)$, $(10\pi/8, -2.8284, -5.8284)$, $(12\pi/8, -3, -4)$

and

$$(14\pi/8, 2.8284, .1716).$$

Figure 6 shows the epitrochoid

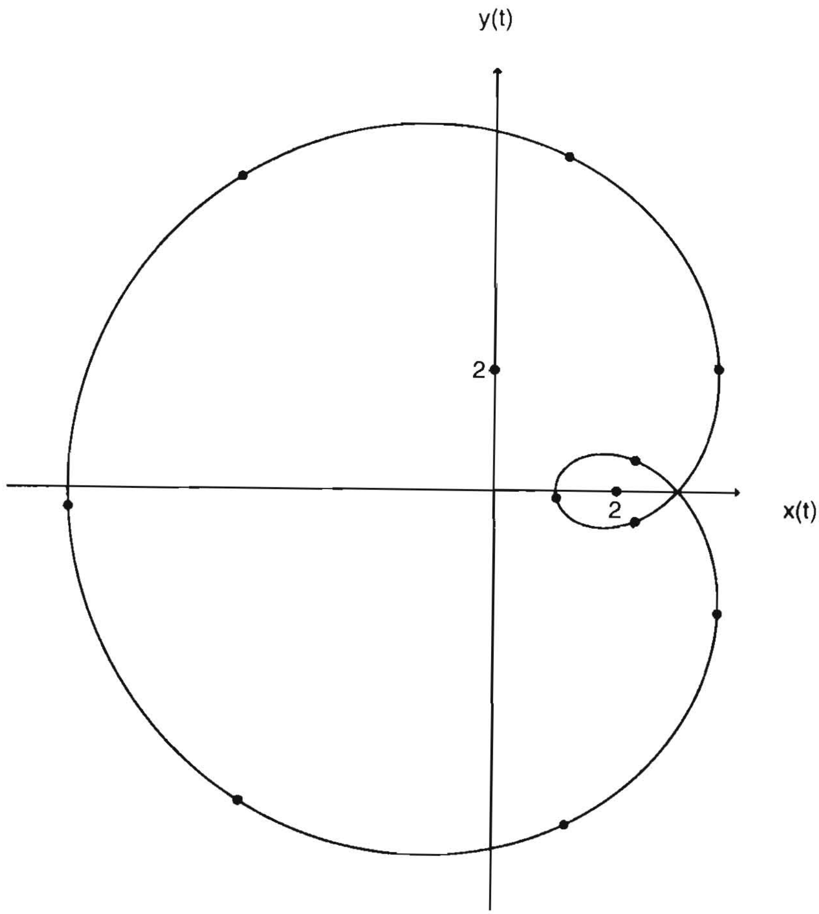

Fig. 6. The epitrochoid.

$$(4 \cos t - 3 \cos 2t, 4 \sin t - 3 \sin 2t)$$

interpolated at the points marked with small circles.

Figure 7 shows the epicycloid

$$(4 \cos t - \cos 4t, 4 \sin t - \sin 4t)$$

interpolated at the points marked with small circles.

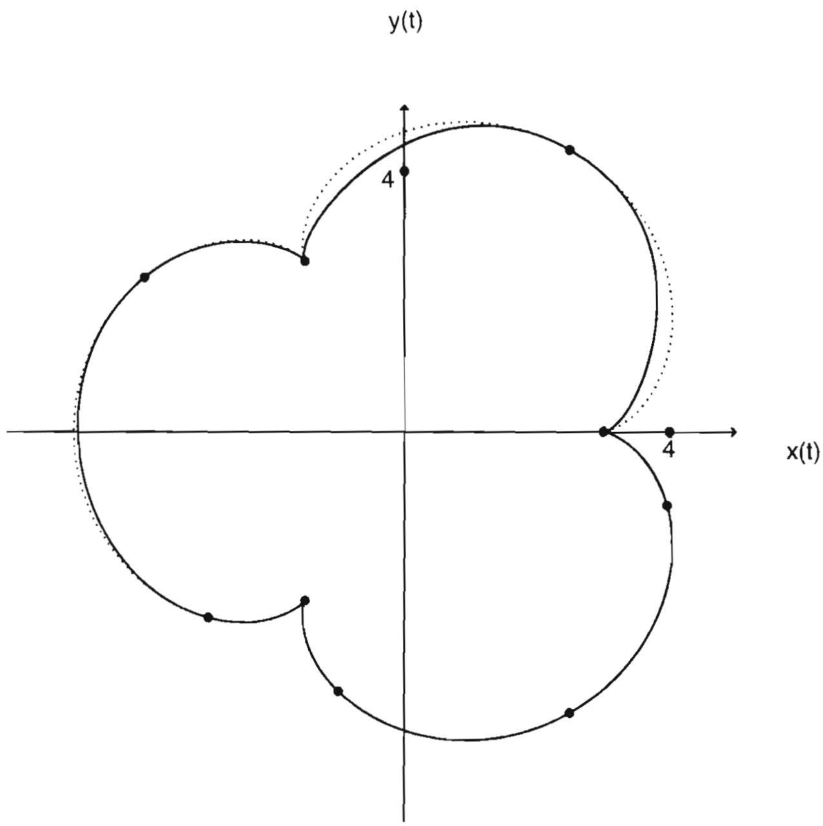

Fig. 7. The epicycloid.

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تقريب جزئي ذو دقة عالية للمنحنيات في المستوى

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نستخدم في التصميم الهندسي بإستخدام الكمبيوتر كثيرات حدود بارميترية للتقريب والاستكمال الداخلي ونمذجة وتمثيل المنحنيات في المستوى . نريد هنا تقريب منحنى منتظم واملس في المستوى باستخدام التقريب الجزئي . نختار التقريب باستخدام كثير حدود جزئي تكعيبي وذلك باستغلال حدية اختيار التابع من اجل الحصول على دقة من الدرجة السادسة وليست دقة من الدرجة الرابعة كما هو الحال باستخدام طرق التقريب المعروفة . كذلك فان تلامس التقريب الجزئي التكعيبي مع المنحنى من الرتبة الثانية مما يعني تساويهما عند نقطتي التقريب وكذلك تساوي المشتقات الأولى والثانية .

ان فكرة هذا البحث مبنية على اساس ملاحظة وجود ستة مجاهيل عند استخدام التقريب الجزئي التكعيبي . نستغل اتصال المنحنى والمشتقة الأولى والمشتقة الثانية عند نقطتي التقريب لايجاد قيم هذه المجاهيل .

ان هذه الطريقة الموصوفة في البحث تختلف عن الطرق التقليدية في انها تنتج نظاماً من المعادلات الغير خطية وهو ما يجعل التعامل معه صعباً وحله

يحتاج إلى تقنية خاصة بشكل عام وكذلك عند تطبيق هذه الطريقة على بعض الأمثلة .

لقد استخدمنا لغة MATLAB في إيجاد القيم الضرورية لتطبيق هذه الطريقة وكذلك في رسم الاشكال والمنحنيات واجراء جميع الحسابات . ان نتائج الامثلة تبين بساطة ايجاد التقريب الجزئي التكعيبي . وكذلك فان الاشكال توضح الفاعلية والنتائج النوعية للطريقة المعروضة في هذا البحث . كما نلاحظ محافظة الطريقة على شكل المنحنى .