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A General Theorem on the Bounds of the Derivative of a Polynomial

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ABSTRACT. Let $\triangle$ be the Unit disk $|z| \leq 1$ in the complex plane **C**. The well known Bernstein's theorem on the bounds of the derivative of an nth degree polynomial f: $\mathbf{C} \rightarrow \mathbf{C}$ states that if $f(\triangle) \subseteq \triangle$ then $|f'(\triangle)| \leq n$ (*i.e.* $|f'(z)| \leq n$ for $|z| \leq 1$). This result was generalized by Szegö and sharpened by Lax under an additional condition. Here, we obtain quite a general theorem that deduces all these results as corollaries and furnishes a chain of interesting new results, some of which offer more general versions (sometimes sharper estimates for $|f'(z)|$) of the theorems of Bernstein, Szegö, and Lax. In fact, we present a unified approach to the basic nature of the problem and its solution underlying Bernstein's theorem and other related Bernstein-type results.

1. Introduction

In the complex plane **C**, we let D(a,r) denote the closed disk with center at the point $a \varepsilon \mathbf{C}$ and radius $r \geq 0$. We agree to write $\triangle_r = D(0,r)$ for $r > 0$ and $\triangle = \triangle_1$ for the unit disk. We denote by π_n the class of all polynomials f: $\mathbf{C} \rightarrow \mathbf{C}$ of degree n and, for a given set $A \subseteq \mathbf{C}$, we write $|A| = \{|a| : a \varepsilon A\}$. By a *circular region* in **C** we mean an open (or closed) set whose boundary is a circle or a straight line. Given $\zeta \; \varepsilon \; \mathbf{C}$ and $f \; \varepsilon \; \pi_n$, the *polar-derivative* $D_\zeta f(z)$ of f with *pole* ζ is defined by

$$(1.1) \qquad D_\zeta f(z) = nf(z) + (\zeta - z) f'(z).$$

The following version of the famous theorem due to Laguerre (1898) (See also Marden 1966) on the zeros of polar-derivatives, will be needed in the sequel.

Theorem 1.1

If all the zeros of a polynomial f: $\mathbf{C} \to \mathbf{C}$ *lie in a circular region* C, *then*

$$D_\zeta f(z) \neq 0 \ \forall \ \zeta, z \notin C.$$

In this paper, we shall obtain a general theorem that deduces the following results as corollaries and furnishes a number of other interesting new results.

Theorem 1.2

(Bernstein 1926). *If* $f \in \pi_n$ *such that* $f(\triangle) \subseteq \triangle$, *then* $f'(\triangle) \subseteq \triangle_n$. *This result is best possible and the extremal polynomial is* $f(z) = \alpha z^n$, *where* $|\alpha| = 1$.

Szegö gave the following generalization of Theorem 1.2.

Theorem 1.3

(Szegö 1928). *If* $f \in \pi_n$ *such that* $|\mathrm{Re}\ f(\triangle)| \leqslant 1$, *then* $f'(\triangle) \subseteq \triangle_n$.

Note that the above theorem is best possible and the extremal polynomial is $f(z) = a + \beta z^n$, where $\mathrm{Re}\ a = 0$ and $|\beta| = 1$.

Theorem 1.4

(Lax 1944). *If* $f \in \pi_n$ *such that* $f(\triangle) \subseteq \triangle$ *and* f *has no zeros in* $|z| < 1$, *then* $f'(\triangle) \subseteq \triangle_{n/2}$.

Observe that Theorem 1.4 gives a sharper bound for $|f'(\triangle)|$, than the one given by Theorem 1.2, under the additional condition on the zeros of f.

2. The Sets with Disk Property

We begin with the following definition.

Definition 2.1.

Given a subset S of **C**, let

$$\mathcal{D}_S = \{D(a,r) : D(a,r) \subseteq S\}.$$

We say that S has the *disk property* (briefly, *d. p.*) if

(2.1) $\rho(S) = \sup\{r: D(a,r) \varepsilon \mathcal{D}_S\} < \infty,$

and call $\rho(S)$ the *core-radius* of S. For each fixed $a \varepsilon S$, we write

(2.2) $\rho_a(S) = \sup\{r: D(a,r) \subseteq S\}.$

In view of (2.1) and (2.2) we clearly have $\rho_a(S) \leqslant \rho(S)$. A point $a_o \varepsilon S$ is called a *core-center* of S if $\rho_{a_o}(S) = \rho(S)$ and the set of all *core-centers* of S will be denoted by *core* (S).

Clearly, a set S with d.p. has interior points if and only if $\rho(S) > 0$. In each of the following examples, the set S has d.p. and core (S) and $\rho(S)$ are as indicated.

Example 2.2

(i) $\rho(S) = r$ and core $(S) = \{a\}$ for $S = D(a,r)$.

(ii) If $S = D(a,r) - \{b\}$, where $b \varepsilon D(a,r)$, then $\rho(S) = (r + |b-a|)/2$ and

$$\text{core } (S) = \begin{cases} \{z: |z-a| = r/2\} & \text{if } b = a, \\ \{a+r_o\, e^{i\theta}\} & \text{if } b \neq a, \end{cases}$$

where $r_o = (|b-a| - r)/2$ and $\theta = \arg(b-a)$.

(iii) If S is any closed or open infinite strip of width 2λ, then $\rho(S) = \lambda$ and core (S) is the set of all points on the axis of the strip. Same is true, in particular, for the strips

$$S = \{z: |\text{Re } z| \leqslant \lambda\} = S_\lambda \text{ (say)},$$

and

$$S = \{z: |\text{Im } z| \leqslant \lambda\} = S^*_\lambda \text{ (say)}.$$

(iv) Consider the strips S_λ and S^*_λ of Example (iii) above. Given $a \varepsilon S_\lambda$ and $b \varepsilon S^*_\lambda$, let L(respectively, L^*) denote the set of all points on the straight line through the point a (respectively, b) drawn parallel to the boundary of S_λ (respectively, S^*_λ). If $S = S_\lambda - L$, then $\rho(S) = (\lambda + |\mathrm{Re}\ a|)/2$ and

$$\mathrm{core}\ (S) = \begin{cases} \{z: |\mathrm{Re}\ z| = \lambda/2\} & \text{if } \mathrm{Re}\ a = 0, \\ \{z: \mathrm{Re}\ z = -\lambda_o\} & \text{if } \mathrm{Re}\ a > 0, \\ \{z: \mathrm{Re}\ z = \lambda_o\} & \text{if } \mathrm{Re}\ a < 0 \end{cases}$$

where $\lambda_o = (\lambda - |\mathrm{Re}\ a|)/2$ so that $0 \leqslant \lambda_o < \lambda/2$.

Similarly, if $S = S^*_\lambda - L^*_\lambda$ then $\rho(S)$ and core (S) are also given by the above expressions provided we replace Re a by Im a and Re z by Im z.

(v) $\rho(S) = (2k)^{\frac{1}{2}}$ and core $(S) = \{0\}$ if

$$S = \{z: |(\mathrm{Re}\ z)\ (\mathrm{Im}\ z) \leqslant k\},\ k > 0.$$

(vi) If $S = \{z: |\mathrm{Re}\ z| \leqslant a,\ |\mathrm{Im}\ z| \leqslant b\}$, $a \geqslant b > 0$, then $\rho(S) = b$ and

$$\mathrm{core}\ (S) = \{z: \mathrm{Im}\ z = 0,\ |\mathrm{Re} z| \leqslant a - b\}.$$

(vii) Given $r > 0$, $0 < \alpha \leqslant \pi/2$ and $0 \leqslant \beta \leqslant \pi/2$, let

$$S = \{z: |z| \leqslant r,\ \beta - \alpha \leqslant \arg z \leqslant \beta + \alpha\} \cup \{0\}.$$

Then $\rho(S) = r \sin \alpha/(1+\sin \alpha)$ and core $(S) = \{z_o\}$, where

$$z_o = re^{i\beta}/(1+\sin \alpha).$$

(viii) If S is the set of points on or inside an ellipse with lengths of axes 2a and 2b, then $\rho(S) = \min\ \{a,b\}$ and core (S) is the center of the ellipse.

(ix) Define a real-valued function $h: [0, +\infty) \to [0,2)$ by $h(x) = 2x/(1+x)$ and let

$$S = \{z: \mathrm{Re}\ z \geqslant 0,\ 0 \leqslant \mathrm{Im}\ z \leqslant h\ (\mathrm{Re}\ z)\}.$$

Then S has d.p. with $\rho(S) = 1$ and core $(S) = \phi$.

Above examples illustrate fairly well that the class of sets having d.p. and a nonempty core is abundant. For unbounded sets with d.p., core (S) may or may not be empty (see Examples 2.2 (iii)-(v) and (ix)). Nevertheless, for bounded sets we have the following result.

Theorem 2.3

If S *is a nonempty bounded subset* of **C**, *then* S *has d.p. and core* $(S) \neq \phi$.

Proof. The first part follows trivially from the boundedness of S. To prove the second part, let without loss of generality $\rho(S) > 0$ and

$$S_n = \{a \varepsilon S: \rho_a(S) \geqslant \rho(S) - 1/n\}, \; n = 1,2,3, \ldots$$

Then we observe the following:

(i) Each $S_n \neq \phi$ by the definition of $\rho(S)$.

(ii) Each S_n is bounded in view of the boundedness of S.

(iii) $S_{n+1} \subseteq S_n$ for every $n \geqslant 1$ (trivial).

(iv) Each S_n is closed. To see this pick any n and fix it. Take any sequence $a_i \; \varepsilon \; S_n$ such that $a_i \rightarrow a_o$ and consider an arbitrary point x such that $|x - a_o| < \rho(S) - 1/n$. Then there exists a positive integer m such that

$$|a_m - a_o| < \rho(S) - 1/n - |x - a_o|.$$

Therefore,

$$|a_m - x| \leqslant |a_m - a_o| + |a_o - x| < \rho(S) - 1/n$$

and so $x \varepsilon S$ (since $a_m \, \varepsilon S_n$). Thus, the interior of the disk $D(a_o, \rho(S) - 1/n)$ lies in S and, by (2.2), $\rho_{a_o} \geqslant \rho(S) - 1/n$.

That is, $a_o \varepsilon S_n$ and S_n is closed.

(v) core $(S) = \bigcap_{n \geqslant 1} S_n$ (easy).

From observations (i) – (iv), the sets S_n, $n \geqslant 1$, from a decreasing sequence of nonempty sets, and hence $\bigcap_{n \geqslant 1} S_n \neq \phi$. Now, the proof is complete in view of the observation (v).

3. The Main Result

We state and prove the following main result of this paper which generalizes a number of known theorems and furnishes some interesting new results with sharper bounds.

Theorem 3.1

If S *has d.p. and* $f \varepsilon \pi_n$ *such that* $f(\triangle_r) \subseteq S$, $r > 0$, *then* $f'(\triangle_r) \subseteq \triangle_{n\rho/r}$, *where* $\rho = \rho(S)$. *This result is best possible and the extremal polynomial is* $f(z) = a+\beta\rho z^n/r^n$, *where* $|\beta| = 1$ *and* $a \ \varepsilon$ core (S), *provided* S *is closed and core* (S) $\neq \phi$.

(See Marden M., § 13, exercise 15).

Proof. Given $w \notin S$, the polynomial $g(z) = f(z)-w$ has all its zeros in the circular region $\mathbf{C}-\triangle_r$ (since $f(\triangle_r) \subseteq S$). By Theorem 1.1, $D_\zeta g(z) \neq 0$ for any $\zeta, z \varepsilon \triangle_r$. Since $D_\zeta g(z) = D_\zeta f(z) - nw$ (cf. (1.1)), it therefore follows that $(1/n)D_\zeta f(z)$ does not assume any value outside S for any $\zeta, z \varepsilon \triangle_r$. That is,

(3.1) $\quad (1/n)D_\zeta f(z) \varepsilon S \ \forall \ \zeta, z \varepsilon \triangle_r.$

Consequently, for each $z \varepsilon \triangle_r$, the point

(3.2) $\quad \zeta f'(z)/n + w^* \ \varepsilon S \ \forall \ \zeta \varepsilon \ \triangle_r,$

where $w^* = f(z)-zf'(z)/n \ \varepsilon S$ for all $z \varepsilon \triangle_r$ (put $\zeta = 0$ in (3.1)). Since $\{\zeta f'(z)/n: \zeta \varepsilon \triangle_r\} = D(0, r\ |f'(z)|\ /n)$ for all $z \varepsilon \triangle_r$,

(3.2) implies that

$$D(0,r\ |f'(z)|/n) + w^* = D(w^*,\ r|f'(z)|/n) \subseteq S \ \forall \ z \varepsilon \triangle_r.$$

From this and the hypotheses on S we conclude that $r|f'(z)|/n \leqslant \rho$ for all $z \varepsilon \triangle_r$. That is, $|f'(\triangle_r)| \leqslant n\rho/r$ and the first statement of the theorem is established.

Regarding the second statement of the theorem, observe that the interior of $D(a,\rho)$ lies in S (since $a \varepsilon$ core(S)) and so $D(a,\rho) \subseteq S$ (since S is closed). For the polynomial f considered, we see that $|f(z)-a| \leqslant \rho$ for $|z| \leqslant r$ and so $f(\triangle_r) \subseteq S$. Clearly, $|f'(z)| = n\rho/r$ for $|z| = r$, and the proof of the theorem is complete.

The above theorem deduces Bernstein's Theorem 1.2 on taking $r = 1$ and $S = \triangle = \triangle_1$, so that $\rho(S) = 1$ and core $(S) = \{0\}$ by Example 2.2(i). Similarly, it deduces Szegö's Theorem 1.3 when we take $r = 1$ and $S = \{w:|\mathrm{Re}\ w| \leqslant 1\}$, so that

$\triangle_1 = \triangle$ and $\rho(S) = 1$ as observed in Example 2.2(iii). More generally, various other possibilities for S as disks of radius s(in particular, say, $S = \triangle_s$) in the above theorem provides the estimate (since $\rho(S) = s$)

(3.3) $f'(\triangle_r) \subseteq \triangle_{ns/r}$.

Similarly, if S is an arbitrary strip of width 2λ, as in Example 2.2(iii), then the above theorem offers the estimate

(3.4) $f'(\triangle_r) \subseteq \triangle_{n\lambda/r}$.

Note that the estimates for $|f'(z)|$ in the theorems due to Bernstein and to Szegö are the special cases $s = r = \lambda = 1$ of the estimates in (3.3) and (3.4).

Next, we employ Theorem 3.1 to get a more general form of Lax's Theorem 1.4 and to obtain a sharper bound for $f'(z)$ than the one given by Szegö's Theorem 1.3 under an additional condition. In the remainder of this section, Int A denotes the interior of a subset A of **C**.

Theorem 3.2

Let $f \varepsilon \pi_n$ *such that* $f(\triangle_r) \subseteq \triangle_s$ $(r,s > 0)$. *If* $a \varepsilon \triangle_s$ *such that* $f(z) \neq a$ *for all* $z \varepsilon$ Int $\triangle_r$, *then* $f'(\triangle_r) \subseteq \triangle_R$, *where* $R = n(s+|a|)/2r < ns/r$ *if* $a\varepsilon$ Int $\triangle_s$.

Proof. To prove the theorem, if suffices to show that $|f'(z_o)| \leqslant R$ for $|z_o| = r$ (use maximum-modulus principle for f'). To this end, we proceed as follows: For each $0 \neq |z| < r$, we apply Example 2.2(ii) and Theorem 3.1 with $\triangle_r$ replaced by $\triangle_{|z|}$ and obtain

$$f'(\triangle_{|z|})\ \lambda \subseteq \triangle_{n(s\ +\ |a|)/2|z|}.$$

In particular,

(3.5) $|f'(z)| \leqslant n(s + |a|)/2|z|$.

If z is made to approach z_o along any path lying completely in Int $\triangle_r$, the continuity of f' and the inequality (3.5) give

$$|f'(z_o)| = \lim_{z \to z_o} |f(z)| \leqslant \lim_{z \to z_o} \frac{n(s+|a|)}{2|z|} = R,$$

and the theorem is established.

For $r = s = 1$ and $a = 0$, Theorem 3.2 is essentially Theorem 1.4 due to Lax (1944).

Theorem 3.3

Let S_λ *denote the strip of Example* 2.2 (iii). *Let* $f \varepsilon \pi_n$ *such that* $f(\triangle_r) \subseteq S_\lambda$. *If* $a \varepsilon S_\lambda$ *such that* $\text{Re} f(z) \neq \text{Re}\, a$ *for all* $z \varepsilon \text{Int} \triangle_r$, *then* $f'(\triangle_r) \subseteq \triangle_\mu$, *where* $\mu = n(\lambda + |\text{Re}\, a|)/2r < n\lambda/r$ *if* $a \varepsilon \text{Int}\, S_\lambda$.

Proof. The proof is based on the technique employed in the proof of Theorem 3.2 (use Example 2.2(iv) in place of Example 2.2(ii)).

Remark 3.4

For $r = \lambda = 1$ and $a = 0$, Theorem 3.3 improves upon the bound for $|f'(z)|$ in Szegö's Theorem 1.3 under an additional condition on the zeros of $\text{Re} f(z)$, in the same manner as Lax's Theorem 1.4 improves upon the bound in Bernstein's Theorem 1.2 under an additional condition on the zeros of f.

Thus for, we have used Theorem 3.1 in obtaining a chain of new results which provide more general versions of the known theorems due to Bernstein (1926), Szegö (1928), and to Lax (1944), but all rallying around disks and strips (only particular instances of sets S with d.p.).

Let us remark that Theorem 3.1 can not be viewed in isolation just as another result contributed to the family of Bernstein-type problems, contrary to what has been the case with existing results. In fact, the general character of Theorem 3.1 exhibits a unified approach to the basic nature of the problem (and its solution) underlying Bernstein's theorem or other related Bernstein-type results due to Szegö (1928), Lax or (possibly) others. In this light, all such results may now be seen as individual reflections of the general character of the sets S used in Theorem 3.1. It is only a matter of picking a right S (whose $\rho(S)$ can be determined) to contribute a new result to the echelon of Bernstein's Theorem. It would neither be worthwhile nor productive to take up such an endless task. However, we shall collect only two new interesting results as demonstration in the following remark.

Remark 3.5

Under the notations and hypotheses of Theorem 3.1, we have the following *results*:

(i) If S is the set of all points common to the disk $\triangle_s$ and a closed sector of angle 2α ($< \pi$) with vertex at origin, then $f'(\triangle_r) \subseteq \triangle_R$, where

$$R = (ns/r) \sin \alpha/(1+\sin\alpha)$$

(See Example 2.2.(vii)). In particular, if S is the closed upper half of the disk $\triangle_s$, then $f'(\triangle_r) \subseteq \triangle_{ns/2r}$.

(ii) If $S = \{z: |(\text{Re } z)(\text{Im} f(z)| \leqslant k\}$, $k > 0$, then

$f'(\triangle_r) \subseteq \triangle_{n(2k)^{\frac{1}{2}}/r}$ by Example 2.2(v).

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قسم الرياضيات ـ جامعة الملك سعود ـ ص. ب : (٢٤٥٥) الرياض ١١٤٥١
المملكة العربية السعودية

نرمز ـ في المستوى المركب C ـ بالرمز Δ_r للقرص المغلق الذي مركزه نقطة الأصل ونصف قطره r ، $r \geqslant 0$، وكذلك نرمز بالرمز π_n لصف كثيرات الحدود $f: C \to C$ من الدرجة النونية. تنص نظرية برينستاين المشهورة والخاصة بحدود مشتقة كثيرة حدود $f \in \pi_n$ على أنه إذا كان $f(\Delta_1) \subset \Delta_1$، فان $|f'(\Delta_1)| \leqslant n$.

ولقد عمم سزيقو ـ في وقت لاحق ـ هذه النظرية، كما حصل لاكس على نتيجة اقوى وذلك تحت شروط اضافية.

نظريـة (سزيقـو) :

إذا كان $f \in \pi_n$ بحيث $|\mathrm{Re}\, f(\Delta_1)| \leqslant 1$، فإن $f'(\Delta_1) \leqslant \Delta_n$.

نظريـة (لاكـس) :

إذا كـان $f \in \pi_n$ بحيث $f(\Delta_1) \subset \Delta_1$ وليس للدالة f أي أصفـار في قرص الـوحدة $|Z| \leqslant 1$، فان $f'(\Delta_1) \leqq \subset \Delta_{n/2}$.

في هـذا البحث ـ وعن طريق تقـديم وتوظيف مـا اصـطلحنـا عـلى تسميتـه بخاصية القرص ـ نحصل على نظرية عامة جداً ليس فقط لكونها تغطّي النظريات

السابقة لكل من برينستاين وسزيقو و لاكس وماشابهها كنتـائج مبـاشرة بـل لانها تـزودنا أيضـاً بسلسلة شيقة أخـرى من النتائـج الجديـدة. وفي الحقيقة، ان هـذه النظرية العامة تزودنا بطريقة موحدة لمعـرفة ومعـالجة طبيعـة هذه المسـألة ومن ثم حلها.