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Saturated p-Normed Algebras and Theorems of Gelfand-Naïmark Type

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ABSTRACT. A definition of saturated uniformly locally A-convex algebras has been given by Cochran (1973). Recently Oudadess (1986) showed that these algebras don't exist and gave another definition. In this paper we extend the notion of saturatedness to the class of p-Normed Algebras and give characterisations of some subclasses. Also some theorems of Gelfand-Naimark type are obtained.

Introduction

Oudadess (1986) studied the saturated Banach algebras. He showed in particular that they aren't any thing else than the uniform ones. In section I we characterize the saturated normed Q-algebras. We show that they are isometrically isomorphic to sub-algebras of an algebra C(K) of continuous functions on a compact K. We also show that the only hermitian saturated normed Q-algebras are the commutative stellar ones. In particular the only hermitian saturated Banach algebras are the commutative $\mathbb{C}^*$-algebras, Zelazko (1965).

In section II we use Le Page (1967) and Cochran (1973) to Prover some theorems of Gelfand-Naïmark type namely, we prove that the commutative $\mathbb{C}^*_p$-algebras (see definition 6) are in fact $\mathbb{C}^*$-algebras. We also prove that the saturated p-Banach algebras are Banach ones, and more generally the saturated p-normed Q-algebras are normed.

Preliminaries

Let E be an algebra over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) and let p be a real number such that $0 < p \leqslant 1$.

Definition 1:

A p-norm in E is a mapping $\|\cdot\|$ from E to $\mathbb{R}_+$ satisfying:

(i) $\|x\| = 0$ if, and only if, $x = 0$

(ii) $\|\lambda x\| = |\lambda|^p \|x\|$, for every $x \in E$ and $\lambda \in \mathbb{K}$

(iii) $\|x+y\| \leqslant \|x\| + \|y\|$, for every x,y in E.

Definition 2:

E is called a p-normed algebra if it is endowed with a p-norm $\|\cdot\|$ such that: $\|xy\| \leqslant \|x\| \cdot \|y\|$ for every x,y in E.

It is said to be a p-Banach algebra if it is complete.

Definition 3:

A unital p-normed algebra is called a Q-algebra if the set of its invertible elements is open.

Definition 4:

A character in E is a multiplicative linear functional. If E is endowed with an involution $*$, it is said to be hermetian if every character X verifies: $X(x^*) = \overline{X(x)}$ for every x in E where $\overline{X(x)}$ is the conjugate complexe of X(x).

I. Saturated p-Normed Algebras

Definition 5:

Let E be a unital p-normed (resp. p-Banach) algebra. We say that E is saturated if there exists an algebra p-norm (resp. complete p-norm) $\|\cdot\|$ defining the topology of E and satisfying:

(1) $\|e\| = 1$, where e is the unit of E.

(2) For every x in E such that $\|x\| = 1$, there exists two characters non identically zero X_0,X of E such that:

$$|X_0(x)| = \sup \{|X(y)|, \|y\| \leqslant 1\}$$

The following proposition characterizes the saturated normed Q-algebras.

Proposition 1

Let $(E, \|\cdot\|)$ be a unital normed Q-algebra. Then E is saturated if, and only if, $\|x\| = \rho(x)$ for every x in E; where $\rho(x)$ is the spectral radius of x *i.e.* $\rho(x) = \sup \{|\lambda| : \lambda \epsilon Spx\}$.

Proof

Suppose that $(E, \|\cdot\|)$ is saturated. Let x be a non zero element of E. We then have $\| \|x\|^{-1} \cdot x\| = 1$, so there exists two characters, non identically zero, X_0, X such that:

$$|X_0 (\|x\|^{-1} \cdot x) | = \sup \{|X(y)|, \|y\| \leqslant 1\}$$

Since $\|e\| = 1$ and $X(e) = 1$, we have $|X_0 (\|x\|^{-1} \cdot x)| \geqslant 1$. Therefore $|X_0(x)| \geqslant \|x\|$. Thus $\rho(x) \geqslant \|x\|$. On the other hand, E is inverse closed in its completion $\hat{E}$ because it is a Q-algebra. Therefore: $\rho(x) = \rho_{\hat{E}}(x) \leqslant \|x\|$ for every x in E.

Conversely, if $\|x\| = 1$ then $\rho(x) = 1$. Since E is a Q-algebra, $\rho(x) = \sup \{|X(x)|, X \epsilon K\} = 1$, where K is the compact space of non identically zero characters. So there exists $X_0 \epsilon K$ such that : $|X_0(x)| = 1$.

Since $|X(y)| \leqslant \|y\|$, for every $X \epsilon K$ and $y \epsilon E$, we have:

$$|X_0(x)| = \sup \{|X(y)|, \|y\| \leqslant 1\}.$$

Thus $(E, \|\cdot\|)$ is saturated.

We obtain, as a corollary, the following result of Oudadess (1986).

Corollary 1:

The saturated Banach algebras are the uniform ones.

We now examine the involutive case.

Proposition 2

Let $(E, \|\cdot\|)$ be a hermitian normed Q-algebra: Then: $(E, \|\cdot\|)$ is saturated if, and only if, $(E, \|\cdot\|)$ is a commutative stellar normed algebra.

Proof

Suppose that $(E, \|\cdot\|)$ is saturated. By proposition 1 we have $\|x\| = \rho(x) = \sup \{|X(x)|, X \in K\}$ for every x in E. But E is hermetian then:

$$\|xx^*\| = \sup \{|X(xx^*)|, X \in K\} = \sup \{|X(x)\overline{X(x)}|, X \in K\} = \|x\|^2$$

Thus $(E, \|\cdot\|)$ is stellar. The commutativity follows from a result of Le Page (1967) since $\|x\| = \rho(x)$, for every x in E.

Conversely, if $(E, \|\cdot\|)$ is a commutative stellar normed algebra, we can show that it is saturated using the fact that it is dense in a commutative $\mathbb{C}^*$-algebra.

Remark:

As an example of a hermitian normed Q-algebra which is not a Banach one, there is the complex algebra $(K(\mathbb{R}))^*$ the unitization of $K(\mathbb{R})$ the algebra of continuous functions over $\mathbb{R}$ with compact support endowed with the supremum norm.

Corollary 2:

Let $(E, \|\cdot\|)$ be a hermitian Banach algebra. Then E is saturated if, and only if, E is a commutative $\mathbb{C}^*$-algebra.

II. Theorems of Gelfand - Naïmark Type

a) Case of normed algebras:

Proposition 3

Let $(E, \|\cdot\|)$ be a saturated normed Q-algebra. Then:

(1) E is isometrically isomorphic to a sub-algebra of $C(K)$; K being the compact space of non identically zero characters.

(2) The Gelfand transform can't be onto unless E is complete.

Proof

Consider the Gelfand transform:

$$(1)\ g : (E, \|\cdot\|) \longrightarrow (C(K), \|\cdot\|_\infty)$$

$$x \longrightarrow g(x) : X \longrightarrow X(x)$$

By proposition 1, we have:

$\|x\| = \rho(x) = \sup\{|X(x)|, X\epsilon K\}) = \|g(x)\|_\infty$, for every x in E.

Thus g is an isometric isomorphism from $(E, \|\cdot\|)$ onto $(g(E), \|\cdot\|_\infty)$.

(2) If $g(E) = C(K)$, then by (1) E and C(k) are isometrically isomorphic; and so $\|\cdot\|$ is complete.

b) Case of Saturated p-Banach and p-normed algebras:

We now examine the structure of the saturated p-Banach algebras. We get:

Proposition 4

Every commutative saturated p-Banach algebra $(E, \|\cdot\|)$ is a Banach one.

Proof

Let x be a non zero element of E. We have $\|\,\|x\|^{-1/p}.x\| = 1$, so there are two characters non identically zero X_0 and X such that:

$$|X_0(\|x\|^{-1/p}.x)| = \sup\{|X(y)|, \|y\| \leqslant 1\} \geqslant 1$$

Therefore $|X_0(x)| \geqslant \|x\|^{1/p}$. Thus $\sup\{|X(x)|^p, X\epsilon K\} \geqslant \|x\|$ for every x in E. On the other hand, Zelazko (1965) has shown that: $\sup\{|X(x)|^p, X\epsilon K\} \leqslant \|x\|$. Hence:

$\|x\| = \sup\{|X(x)|^p, X\epsilon K\} = (\sup\{|X(x), X\epsilon K\})^p$. Thus $\|\cdot\|$ is equivalent to the norm defined by:

$$\|x\|_\infty = \sup\{|X(x)|, X\epsilon K\}$$

More generally, we have the following proposition.

Proposition 5

Let $(E, \|\cdot\|)$ be a commutative saturated p-normed Q-algebra. Then E is normed.

Proof

We shall show that: $\|x\| = \sup \{|X(x)|^p, X \in K\}$ for every x in E.

Since $(E, \|\cdot\|)$ is saturated, we can show as above, that $\sup \{|X(x)|^p, X \in K\} \geqslant \|x\|$ for every x in E.

Conversely, if X is an element of K, then $|X(x)|^p \leqslant \|x\|$ for if $\|x\| < |X(x)|^p$, $\|(X(x))^{-1}.x\| < 1$. But $(e-(X(x))^{-1}x)$ will be invertible, since E is a Q-algebra. Therefore $(X(x)e-x)$ is invertible. This is impossible because $X(x) \in Spx$. Thus: $\sup \{|X(x)|, X \in K\} = \|x\|^{1/p}$, and so $\|\cdot\|$ is equivalent to the algebra norm $\|\cdot\|_\infty$ given by: $\|x\|_\infty = \sup \{|X(x)|, X \in K\}$.

C. The $\mathbb{C}^*_p$-algebras.

By analogy with the $\mathbb{C}^*$-algebras, one can define the $\mathbb{C}^*_p$-algebras as follows:

Definition 6:

An involutive p-Banach algebra $(E, \|\cdot\|)$ is called a $\mathbb{C}^*_p$-algebra if:

$\|xx^*\| = \|x\|^2$ for every x in E.

Examples

1) Every $\mathbb{C}^*$-algebra is a $\mathbb{C}^*_p$-algebra.

2) If $(E, \|\cdot\|)$ is a $\mathbb{C}^*$-algebra and p is a real number such that $0 < p \leqslant 1$, then $(E, \|\cdot\|^p)$ is a $\mathbb{C}^*_p$-algebra.

We show that in the commutative case every $\mathbb{C}^*_p$-algebra is in fact a $\mathbb{C}^*$-algebra.

We shall need some results of Zelazko (1965) which we give in the following proposition.

Proposition 6

Let $(E, \|\cdot\|)$ be a commutative unital p-Banach algebra. Then:

(1) The mapping defined by $= \|x\| = \operatorname*{Lim}_{n \to +\infty} \|x^n\|^{1/n}$ is an algebra p-semi-norm such that $\|x\|_s \leqslant \|x\|$ for every x in E

(2) $\|x\|_s = \sup\{|X(x)|, X \in K\}$, $x \in E$.

We now obtain

Theorem

Let $(E, \|\cdot\|)$ be a unital commutative $\mathbb{C}^*_p$-algebra. Then $\|\cdot\|$ is equivalent to a $\mathbb{C}^*$-algebra norm.

Proof

We shall show that $\|x\|_s = \|x\|$ for every x.

Let h be a hermitian element in $E(h^* = h)$. We have $\|h^2\| = \|h \cdot h^*\| = \|h\|^2$. So for every positive integer n, we have: $\|h\| = \|h^{2^n}\|^{2^{-n}}$. Therefore $\|h\|_s = \|h\|$. For an arbitrary element x, xx^* is a hermetian one so:

$$\|x\|^2 = \|xx^*\| = \|xx^*\|_s \leq \|x\| \quad \|x^*\|_s \leq \|x\|_s \, \|x^*\|$$

Whence $\|x\|^2 \leq \|x\|_s \cdot \|x\|$, since $\|x\| = \|x^*\|$. Thus $\|x\| \leq \|x\|_s$. The other inequality is always satisfied.

Consider the mapping $|\cdot|$ defined by: $|x| = \|x\|_s^{1/p}$. Then (2) of proposition 6 ensures that it is an algebra semi-norm. Since $\|x\| = \|x\|_s$, $x \in E$, we have:

$|x| = \|x\|^{1/p}$ for every x in E. Hence $|\cdot|$ is a complete algebra norm equivalent to $\|\cdot\|$.

Moreover, $|xx^*| = \|xx^*\|^{1/p} = (\|x\|^2)^{1/p} = |x|^2$. Thus $|\cdot|$ is a $\mathbb{C}^*$-algebra norm. This completes the proof.

Remarks:

1) There is no commutative unital $\mathbb{C}^*_p$-algebra which is not a $\mathbb{C}^*$-algebra.

2) By adjoining a unit, we can see that the last remark is also true in the non unital case.

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الجبور P ـ المُنظمة المشبعة ومبرهنات على نمط مبرهنات جلِفاند ـ نَيْمارك

عبدالوهاب بضعة و محمَّد أُدادس

قسم الرياضيات ـ المدرسة العليا للأساتذة ـ التّقدُّم ـ ص . ب : ٥١١٨ ـ الرِّباط ـ المغرب

الهدف من هذا البحث هو :

I ـ تعريف الجبور P ـ المُنظمة ودراسة خواص بعض صفوفها الجزئية .

II ـ إثبات مبرهنات من نمط مبرهنات نَيْمارك ـ جلفاند .

وقد كان تسلسل العمل في البحث على النحو التالي :

I ـ تمَّ إيراد خمسة تعاريف عن

١ ـ P ـ نظيم في جبر E هو تطبيق $\|\cdot\|$ من E إلى $\mathbb{R}^+$ يحقق شروطاً ثلاثة .

٢ ـ نقول عن E أنه جبر P ـ مُنظم إذا كان مُزوَّد بِـ P ـ نظيم $\|\cdot\|$ وبحيث يتحقّق . $\|xy\| \leq \|x\| \cdot \|y\|$ من أجل كل x ، y من E .

٣ ـ نقول عن جبر P ـ مُنظم واحدي أنه Q ـ جبر إذا كانت مجموعة عناصره القابلة للقلب مجموعة مفتوحة .

٤ ـ كل مميِّز في الجبر E هو تابع خطي ضربي .

٥ ـ ليكن E جبر، P ـ مُنظم واحدي (أو P ـ باناخ). نقول عن E أنه

مُشبع ، إذا وُجِد P ـ نظيم جبري (أو P ـ نظيم تام) $\|\cdot\|$ يُعرّف تبولـوجيا E ويحقق ما يلي :

١ ـ $\|e\| = 1$ حيث e هو عنصر واحدة الجبر E .

٢ ـ من أجل كل عنصر x من E حيث $\|x\| = 1$ ، يُوجد مميّـزين x ، x_0 مغايران للصفر ويحققان .

$$|X_0(x)| = \sup \{|X(y)|, \|y\| \leq 1\}$$

ثم بُرهِنت القضيتين التاليتين :

قضيّـة (١) :

ليكن $(E, \|\cdot\|)$، Q ـ جبر مُنظم واحدي . الشرط اللاَّزم والكـافي لكي يكون E مشبعاً هو أن يكون $\rho(x) = \|x\|$ من أجل كـل x من E ، حيث $\rho(x)$ هي نصف القطر الطيفي لـِ x .

نتيجـة (١) :

جبور باناخ المشبعة هي جبور منتظمة .

قضيّـة (٢) :

ليكن $(E, \|\cdot\|)$، Q ـ جـبر مُنظم هـرميتي . الشرط اللاَّزم والكـافي لكي يكون $(E, \|\cdot\|)$ جبراً مشبعاً هو أن يكون $(E, \|\cdot\|)$ جبراً مُنظماً نجمياً تبديلياً .

نتيجـة (٢) :

ليكن $(E, \|\cdot\|)$ جبر باناخ هرميتي . الشرط الـلاَّزم والكافي لكي يكـون E جبراً مشبعاً هو أن يكون E، $\mathbb{C}^*$ ـ جبر تبديلي .

II - تمَّ في هـذا القسم إيراد التعـريف التـالي «نقـول عن جـبر P - بـانـاخ ملتف (المرموز له بـ$(E, \|\cdot\|)$) أنه $\mathbb{C}_p^*$- جبر إذا حقق $\|XX^*\| = \|X\|^2$ من أجل كل x من E». وتمَّ بعدها إثبات بعض النتائج من نمط مبرهنات جلفاند - نَيْمارك. وقد تمّ التمييز بين حالتين.

الحالـة الأولـى :

وهي الحالة التي تكون فيها الجبور مُنظمة. وفيها بُرهنت القضية التالية.

قضيـة (٣) :

ليكن $(E, \|\cdot\|)$، Q - جبر مُنظم مشبع. في هذا الجبر يتحقق ما يلي :

١ - E هو إيزومورف تقايسي مع جبر جزئي من C(K)، حيث K هو فضـاء متراص مميزه مغاير للصفر.

٢ - لا يمكن لتحويل جِلفاند أن يكون غامراً إلاَّ إذا كان الجبر E تام.

الحالـة الثانيـة :

حالة الجبور P - المنظمة والـ P - بانـاخ المشبعة. حيث تمَّ بفتحص ودراسـة جبور P - باناخ برهان التالي :

قضيـة (٤) :

كل جبر P - باناخ مُشبع تبديلي $(E, \|\cdot\|)$ هو جبر باناخ.

قضيـة (٥) :

إذا كان $(E,\|\cdot\|)$، Q - جبر، P - مُنظم مشبع تبديلي. فإن E يكون منظم.

قضيـة (٦) :

ليكن $(E,\|\cdot\|)$ جبر P ـ باناخ واحدي تبديلي، عندها يتحقق ما يلي :

١ ـ التطبيق المُعرَّف بـِ $\|x\|_s = \lim\limits_{n\to+\infty} \|x^n\|^{1/n}$ هـو جـبر، P - نصف نـظيم بحيث

$\|x\|_s \leq \|x\|$ من أجل كل x في E

٢ ـ $\|x\|_s = \sup \{|X(x)|, X \in K\}, x \in E.$

مبرهنـة :

ليكن $(E,\|\cdot\|)$، $\mathbb{C}^*_p$ ـ جبر تبديلي واحدي، عندئذٍ يكون $\|\cdot\|$ مكافىء لنظيم $\mathbb{C}^*$ـ جبر.