Arab Gulf J. Scient. Res., **10**(3), pp. 1-15 (1992)

Fuzzy Algebraic Structures of Universal Algebras

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ABSTRACT. The aim of this paper is to give a systematic study of fuzzy subalgebras of a universal algebra. We introduce the notions of fuzzy subalgebra generated by a fuzzy set, and fuzzy congruences on an algebra. We also introduce a fuzzy congruence class and fuzzy quotient algebra and prove that fuzzy quotient algebras are isomorphic to a certain ordinary quotient algebra in a natural way.

1. Introduction

In recent years much attention have been given to generalizing the classical notions of groups, rings, vector spaces algebras over a field, and lattices in the context of fuzzy sets (Das 1981, Dixit *et al.* 1990, Lubczonok 1990, Muganda 1991, Murali 1991a, 1991b, and Nanda 1986) Gerla and Tortora (1985) introduced the notion of fuzzy subalgebra of a universal algebra. In this paper we continue investigating the notion of fuzzy subalgebra of a universal algebra providing several results. We prove that the homomorphic image and pre-image of a fuzzy subalgebra is again a fuzzy subalgebra, and the direct product of fuzzy subalgebras is a fuzzy subalgebra. We introduce the notion of fuzzy subalgebra generated by a fuzzy set and give a method to construct a fuzzy subalgebra from a fuzzy subset of finite range. Some characterization theorems are given. We notice that the collection of all fuzzy subalgebras of a given algebra form a lattice. Finally, the

Keywords: Universal algebra, fuzzy subalgebra, direct product of fuzzy subalgebras, fuzzy congruence relation, fuzzy congruence class, fuzzy quotient algebra.

notion of fuzzy congruence on an algebra, fuzzy congruence classes, fuzzy quotient algebras are introduced and prove that fuzzy quotient algebras are isomorphic to a certain ordinary quotient algebras in a natural way.

2. Some Basic Results

Throughout this paper, I denotes the unit interval [0,1]. All fuzzy subsets of a nonempty set X are maps μ: X $\rightarrow$ I, and all fuzzy relations on X are maps θ: XxX $\rightarrow$ I. For elements a,b, $\in$ I,a $\wedge$ b = min {a, b} and avb = max {a, b}. For fuzzy subsets μ and υ of X, $\mu \leq \upsilon$ means $\mu(x) \leq \upsilon(x)$ for all x $\in$ X. A fuzzy relation θ on X is called fuzzy equivalence relation if $\theta(x,x) = 1$, $\theta(x,y) = \theta(y,x)$ and $\theta(x,y) \geq \theta(x,z) \wedge \theta(z,y)$ for all x, y, z $\in$ X. We denote the characteristic function of X by 1_X.

Definition 2.1 (Grätzer 1979) An algebra (or universal algebra) is a triple **A** = < A, C, F > such that A is a nonempty set and C is a subset of A (possibly empty) whose elements are called constant of A, and F is a set of finitary operations on A (*i.e.* an element of F is a map from A^n to A for a suitable positive integer n). If B is a none-empty subset of A, then **B** = < B, C, F > is called a subalgebra of A if C $\subseteq$ B and for every n-ary function f in F and n elements b_1, b_2,...b_n, in B, f $(b_1, b_2,..., b_n) \in$ B.

Definition 2.2 (Gerla and Tortora 1985) Let **A** be an algebra. A fuzzy subalgebra of **A** is a fuzzy subset μ of A such that

(i) $\mu(c) \geq \mu(x)$ for all c $\in$ C, x $\in$ A,

(ii) For every n-ary f $\in$ F and any elements $a_1, a_2,...,a_n$ in A

$$\mu((a_1, a_2,..., a_n)) \geq \mu(a_1) \wedge \mu(a_2) \wedge .... \wedge \mu(a_n).$$

The notions of fuzzy subgroups, fuzzy subrings, fuzzy subfields, and fuzzy sublattices are easily shown to be particular cases of fuzzy subalgebras.

Proposition 2.3. If **A** is an algebra, then **B** ($\emptyset \neq B \subseteq A$) is a subalgebra of **A** iff 1_B is a fuzzy subalgebra of **A.**

Proof. Assume that 1_B is a fuzzy subalgebra of **A.**Let f be any n-ary function in F and $b_1, b_2,..., b_n \in$ B. Since $b_i \in$ B for all

$i = 1,2,...,n$, $1_B(b_i) = 1$ for all $i = 1,2,...,n$. Thus,

$1_B(f(b_1, b_2,... b_n)) \geq 1_B(b_1) \wedge 1_B(b_2) \wedge ... \wedge 1_B(b_n) = 1$,so

$1_B(f(b_1, b_2, \ldots b_n)) = 1$ and hence $f(b_1, b_2, \ldots b_n) \in B$. For the converse, assume that **B** is a subalgebra of **A**. Since $C \subseteq B$, we have $1_B(c) = 1$ for all $c \in C$, and hence $1_B(c) \geq 1_B(x)$ for all $c \in C$, $x \in A$.

Now, let f be n-ary in F and $a_1, a_2, \ldots, a_n \in A$. We must show that

$$1_B(f(a_1, a_2, \ldots a_n)) \geq 1_B(a_1) \wedge 1_B(a_2) \wedge \ldots \wedge 1_B(a_n) \quad (1)$$

We have the following two cases:

(i) $a_i \in B$ for all $i = 1,2, \ldots, n$. In this case $f(a_1, a_2, \ldots, a_n) \in B$, so that,

$1_B(f(a_1, a_2, \ldots, a_n)) = 1$ and (1) holds.

(ii) At least one $a_j \notin B$. In this case $1_B(a_j) = 0$, and hence

$1_B(a_1) \wedge 1_B(a_2) \wedge \ldots \wedge 1_B(a_n) = 0$ which shows that (1) also hold.

Definition 2.4 (Sherwood 1983) Let μ be a fuzzy subset of **A.** For all $t \in [0,1]$, the set $\mu_t = \{ x \in A \mid \mu(x) \geq t \}$ is called level subset of μ.

Proposition 2.5 Let **A** be an algebra and μ be a fuzzy subalgebra of **A.** Then for all $t \in I$, μ_t is a subalgebra of **A** provided $\mu_t \neq \emptyset$.

Proof. Since $\mu_t \neq \emptyset$, there is an element $x \in A$ such that $\mu(x) \geq t$. But as $\mu(c) \geq \mu(x)$ for all $c \in C$, $x \in A$, we have $\mu(c) \geq t$ for all $c \in C$ and hence $c \in \mu_t$ for all $c \in C$. Moreover, if f is n-ary in F and $a_1, a_2, \ldots, a_n \in \mu_t$, then $\mu(a_i) \geq t$ for all $i = 1,2, \ldots, n$, and so

$\mu(f(a_1 \cdot a_2, \ldots, a_n)) \geq \mu(a_1) \wedge \mu(a_2) \wedge \ldots \wedge \mu(a_n) \geq t$, hence $f(a_1, a_2, \ldots, a_n) \in \mu_t$ and so μ_t is a subalgebra of A.

Proposition 2.6 Let **A** be an algebra and μ a fuzzy subset of A such that μ_t is a subalgebra of **A** for all $t \in I$. Then μ is a fuzzy subalgebra of **A.**

Proof. Since μ_t is a subalgebra of **A,** $\mu(c) = 1$ for all $c \in C$. Now let f be n-ary in F, and $a_1, a_2, \ldots, a_n$ in A with $\mu(a_i) = t_i$. Then $a_i \in \mu_{t_i}$ for all

$i = 1,2, \ldots, n$. Assume $t_1 < t_2 < \ldots < t_n$. Then it follows that

$\mu_{t_n} \subseteq \mu_{t_{n-1}} \subseteq \ldots \subseteq \mu_{t_1}$. Hence $a_i \in \mu_{t_1}$ for all $i = 1,2, \ldots n$. As μ_{t_1} is subalgebra of **A,** we have $f(a_1, a_2, \ldots, a_n) \in \mu_{t_1}$. Hence,

μ $(f(a_1, a_2,\ldots,a_n)) \geq t_1 = \mu(a_1) \wedge \mu(a_2) \wedge\ldots\wedge \mu(a_n)$. Therefore, μ is a fuzzy subalgebra of **A**.

3. Homomorphisms and Direct Products

Definition 3.1 (Grätzer 1979) Suppose that $\mathbf{A} = < A,C,F >$ and $\mathbf{B} = < B, C', F' >$ are two algebras of the same type. A mapping α : $A \rightarrow B$ is called a homomorphism from **A** to **B** if, for every f n-ary in F there exists f′ n-ary on α (A), $f' \in F'$ such that

$$\alpha(f(a_1, a_2,\ldots,a_n)) = f'(\alpha(a_1), \alpha(a_2),\ldots \alpha(a_n))$$

for each sequence $a_1, a_2,\ldots, a_n$ from **A.** and if, Moreover,

for all $c \in C$, we have $\alpha(c) \in C'$

Definition 3.2 (Nanda 1986) **A** fuzzy subset μ of a set A is said to have the sup property if, for any subset $B \subseteq A$, there exists $b_o \in B$ such that

$$\mu(b_o) = \bigvee_{b \in B} \mu(b).$$

Theorem 3.3 Let $\alpha : \mathbf{A} \rightarrow \alpha(\mathbf{A})$ be a homomorphism.

(i) If υ is a fuzzy subalgebra of **A** and υ has the sup property, then the image μ of υ is a fuzzy subalgebra of α (**A**), where μ is defined by

$$\mu(y) = \bigvee_{x \in \alpha^{-1}(y)} \upsilon(x) \text{ for all } y \in \alpha(A).$$

(ii) If μ is a fuzzy subalgebra of α (**A**), then the preimage υ of μ is a fuzzy subalgebra of **A**, where υ is defined by $\upsilon = \mu \circ \alpha$.

Proof.

(i) Notice first that if $c \in C$ and $\alpha(a) \in \alpha(A)$, then let $b \in \alpha^{-1}(\alpha(a))$ such that

$$\upsilon(b) = \bigvee_{x \in \alpha^{-1}(\alpha(a))} \upsilon(x)$$

Then,

$$\mu(\alpha(c)) = \bigvee_{x \in \alpha^{-1}(\alpha(c))} \upsilon(x) \geq \upsilon(b) = \mu(\alpha(a))$$

Now, let $\alpha(a_1), \alpha(a_2), \ldots. \alpha(a_n)$ be in $\alpha(A)$ and let f' be an n-ary function on $\alpha(A)$. Let f be the n-ary function on A such that

$$f'(\alpha(a_1), \ldots, \alpha(a_n)) = \alpha(f(a_1, \ldots, a_n))$$

Since υ has the sup property, we can find

$$b_1 \in \alpha^{-1}(\alpha(a_1)), \ldots, b_n \in \alpha^{-1}(\alpha(a_n))$$

such that

$$\upsilon(b_1) = \bigvee_{x \in \alpha^{-1}(\alpha(a_1))} \upsilon(x), \ldots, \upsilon(b_n) = \bigvee_{x \in \alpha^{-1}(\alpha(a_n))}$$

Hence,

$$\mu\left(f'(\alpha(a_1), \ldots, \alpha(a_n))\right) = \bigvee_{z \in \alpha^{-1}(f'(\alpha(a_1), \ldots, \alpha(a_n)))} \upsilon(z)$$

$$= \bigvee_{z \in \alpha^{-1}(f'(\alpha(b_1), \ldots, \alpha(b_n)))} \upsilon(z)$$

$$= \bigvee_{z \in \alpha^{-1}(\alpha'(f(b_1), \ldots, (b_n)))} \upsilon(z)$$

$$\geq \upsilon(f(b_1, \ldots, b_n))$$

$$\geq \upsilon(b_1) \wedge \upsilon(b_2) \wedge \ldots \wedge \upsilon(b_n)$$

$$= \mu(\alpha(a_1)) \wedge \mu(\alpha(a_2)) \wedge \ldots \wedge \mu(\alpha(a_n)).$$

Therefore, μ is a fuzzy subalgebra of $\alpha(\mathbf{A})$.

(ii) Let $a_1, a_2, \ldots, a_n \in A$, f an n-ary function of A and f' an n-ary function on $\alpha(A)$.

Then

$$\upsilon(f(a_1, a_2, \ldots, a_n)) = \mu(\alpha(f(a_1, a_2, \ldots, a_n)))$$

$$= \mu(f'(\alpha(a_1), \alpha(a_2), \ldots, \alpha(a_n)))$$

$$\geq \mu(\alpha(a_1)) \wedge \mu(\alpha(a_2)) \wedge \ldots \wedge \mu(\alpha(a_n))$$

$$= \upsilon(a_1) \wedge \upsilon(a_2) \wedge \ldots \wedge \upsilon(a_n).$$

Also, if $c \in C$ and $a \in A$, then $\alpha(a) \in \alpha(A)$ and $\upsilon(c) = \mu(\alpha(c)) \geq \mu(\alpha(a)) = \upsilon(a)$.

Therefore υ is a fuzzy subalgebra of **A.**

We prove now that the direct product of fuzzy subalgebras is again a fuzzy subalgebra.

Theorem 3.4 If μ_i is a fuzzy subalgebra of $\mathbf{A_i}$ for all $i \in J$, then

$\prod_{i \in J} \mu_i$ is a fuzzy subalgebra of $\prod_{i \in J} \mathbf{A}_i$, where $\prod_{i \in J} \mu_i$ is defined as

$$\prod_{i \in J} \mu_i(x) = \bigwedge_{i \in J} \mu_i(x_i) \text{ for all } x = \langle x_i \rangle \in \prod_{i \in J} \mathbf{A}_i.$$

Proof. Let $\mu = \prod_{i \in J} \mu_i$ and $\mathbf{A} = \prod_{i \in J} \mathbf{A_i}$. It is clear that $\mu(c) \geq \mu(x)$ for all $c \in C$

and $x \in A$. Let $x_1 = \langle x_{i1} \rangle, x_2 = \langle x_{i2} \rangle, \ldots, x_n = \langle x_{in} \rangle$ in A and f n-ary in F.

Then

$$\mu(f(x_1, x_2, \ldots, x_n)) = \bigwedge_{i \in J} \mu_i(f(x_{i1}, x_{i2}, \ldots, x_{in}))$$

$$\geq \bigwedge_{i \in J} \mu_i(x_{i1}) \wedge \mu_i(x_{i2}) \wedge \ldots \wedge \mu_i(x_{in})$$

$$\geq \bigwedge_{i \in J} \mu_i(x_{i1}) \wedge \bigwedge_{i \in J} \mu_i(x_{i2}) \wedge \ldots \wedge \bigwedge_{i \in J} \mu_i(x_{in})$$

$$= \mu(x_1) \wedge \mu(x_2) \wedge \ldots \wedge \mu(x_n).$$

Therefore, μ is a fuzzy subalgebra of **A.**

4. Fuzzy Subalgebras Generated by Fuzzy Subsets

Proposition 4.1 If $\{ \mu_i : i \in J \}$ is a nonempty family of fuzzy subalgebras of the algebra A, then $\bigwedge_{i \in J} \mu_i$ is a fuzzy subalgebra of A.

Hence,

$$\mu^* (f(a_1,a_2,\ldots, a_n)) \geq t_j = \mu^* (a_1) \wedge \mu^* (a_2) \wedge \ldots \wedge \mu^* (a_n).$$

The fact that $\mu^* (x) \geq \mu (x)$ for all $x \in A$ follows directly from the definition of μ^*.

The following theorem shows that $\mu^* = \text{Fsg} (\mu)$.

Theorem 4.6 Let υ be a fuzzy subalgebra of **A** such that $\mu \leq \upsilon$, then $\mu^* \leq \upsilon$.

Proof. By Proposition 2.5, $\upsilon_t = \{x \in A : \upsilon (x) \geq t\}$ is a subalgebra of A for all $t \in [0, 1]$. Since $\mu \leq \upsilon$ we have $Y_i \subseteq \upsilon_{t_i}$ for all $1 \leq i \leq k$, hence $B_i \subseteq \upsilon_{t_i}$.

Thus for each $x \in B_i - B_{i-1}$ we have $\mu^* (x) = t_1 \leq \upsilon (x)$. Therefore we conclude that $\mu^* (x) \leq \upsilon (x)$ for all $x \in A$.

Definition 4.7 Let **A** be an algebra, and denote the collection af all fuzzy subalgebras of **A** by Fsub (**A**). Define addition and multiplication on Fsub (A) by

$\mu_1 + \mu_2 = \text{Fsg} (\mu_1 \vee \mu_2)$ and $\mu_1 . \mu_2 = \mu_1 \wedge \mu_2$.

It is evident that the triple $<$ Fsub (**A**), $+,.>$ form a lattice.

5. Fuzzy Congruences and Fuzzy Congruence Classes

Definition 5.1 Let θ be a fuzzy equivalence relation on an algebra **A**. Then θ is a fuzzy congruence on A if for any n-ary $f \in F$ and any elements $a_1,a_2,\ldots, a_n, b_1, b_2,\ldots, b_n$ from **A**

$$\theta (f(a_1, a_2,\ldots, a_n), f(b_1, b_2,\ldots, b_n)) \geq \theta (a_1, b_1) \wedge \theta (a_2, b_2) \wedge \ldots \wedge \theta (a_n, b_n)$$

We denote by FC(A) the collection of all fuzzy congruences on **A.**

Definition 5.2 (Murali 1989) If θ is a fuzzy relation on a set A, then th ordinary relation on A associated with θ, denoted by θ^* is defined as

$$\theta^* = \{ (x,y) : \theta (x,y) = 1\}.$$

Proposition 5.3 (Murali 1989). If θ is fuzzy equivalence relation on a set A, then θ^* is an ordinary equivalence relation on A.

Proposition 5.4 If θ is a fuzzy congruence on the algebra **A**, then θ^* is an ordinary congruence on **A.**

Proof. By Proposition 5.3, θ^* is an equivalence relation on A. Let f be any n-ary in F and $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ in A such that $a_1 \theta^* b_i$ for all $i = 1,2,\ldots,n$. Then,

$\theta(a_i, b_i) = 1$ for all $i = 1,2,\ldots,n$, and

$$\theta(f(a_1, a_2,\ldots,a_n), f(b_1, b_2,\ldots, b_n)) \geq \theta(a_1, b_1) \wedge \theta(a_2, b_2) \wedge \ldots \wedge \theta(a_n, b_n) = 1.$$

Thus, $\theta(f(a_1, a_2,\ldots,a_n), f(b_1, b_2,\ldots,b_n)) = 1$ and so

$f(a_1,a_2,\ldots,a_n) \; \theta^* \; f(b_1,b_2,\ldots,b_n)$.

Definition 5.5 Let **A** be an algebra, $a \in A$, and $\theta \in FC(\mathbf{A})$. The fuzzy congruence class determined by a and θ, denoted by $a\theta$ is the fuzzy subset of A defined by

$$a\theta(b) = \theta(a,b) \text{ for all } b \in A.$$

Definition 5.6 Let $\theta \in FC(\mathbf{A})$ and A/θ be the set of all fuzzy congruence classes, that is $A/\theta = \{a\theta : a \in A\}$. Let f be an n-ary in F.

Then the n-ary **function** f^* on A/θ induced by f is defined as

$$f^*(a_1\theta, a_2\theta,\ldots, a_n\theta) = f(a_1, a_2,\ldots,a_n)\,\theta.$$

Theorem 5.7 f^* is well-defined.

Proof. Let $a_1, a_2,\ldots,a_n, b_1, b_2,\ldots, b_n$ in A such that

$a_1\theta = b_1\theta, a_2\theta = b_2\theta,\ldots,a_n\theta = b_n\theta$. Then

$\theta(a_i, b_i) = a_i\theta(b_i) = b_i\theta(b_i) = \theta(b_i, b_i) = 1$ for all $i = 1,2,\ldots,n$. Thus,

$$\theta(f(a_1, a_2,\ldots,a_n), f(b_1, b_2,\ldots,b_n)) \geq \theta(a_1, b_1) \wedge \theta(a_2, b_2) \wedge \ldots \wedge \theta(a_n,b_n) = 1.$$

So,

$$\theta\ (f(a_1, a_2, \ldots, a_n), f(b_1, b_2, \ldots, b_n)) = 1.$$

Now, for any $x \in A$

$$f(a_1, a_2, \ldots, a_n)\ \theta\ (x) = \theta\ (x, f(a_1, a_2, \ldots, a_n))$$

$$= \theta\ (x, f(a_1, a_2, \ldots, a_n)) \wedge \theta\ (f(a_1, a_2, \ldots a_n), f(b_1, b_2, \ldots, b_n))$$

$$\leq \theta\ (x, f(b_1, b_2, \ldots, b_n))$$

$$= f(b_1, b_2, \ldots, b_n)\ \theta\ (x).$$

Similarly; $f(b_1, b_2, \ldots, b_n)\ \theta\ (x) \leq f(a_1, a_2, \ldots, a_n)\ \theta\ (x)$. Therefore,

$f(a_1, a_2, \ldots, a_n)\ \theta = f(b_1, b_2, \ldots, b_n)\ \theta$ and f^* is well defined.

Definition 5.8 The algebra $< A/\theta, C/\theta, F^* >$ with underlying set A/θ (Set of constanb $C/\theta = \{c\theta, c \in C\}$ and set of operations $F^* = \{ f^* : f \in F \}$ is called the quotient algebra of $\mathbf{A} = <(A, C, F >|)$ **A** algebra of **A** by θ.

Definition 5.9 Let θ be a fuzzy congruence relation on A. Define the quotient fuzzy relation $\overline{\theta}$ on $\mathbf{A}/\theta$ as $\overline{\theta}\ (a\theta, b\theta) = \theta(a,b)$ for all $a, b \in A$.
The following proposition is straightforward.

Proposition 5.10 $\overline{\theta} \in FC\ (\mathbf{A}/\theta)$.

Definition 5.11 Let $\theta \in FC\ (\mathbf{A})$ and $\overline{\varphi}$ is a fuzzy relation on A/θ.

Define the fuzzy relation φ on A by $\varphi\ (a,b) = \overline{\varphi}\ (a\theta, b\theta)$ for all $a, b \in A$.

The following proposition is evident.

Proposition 5.12 If $\overline{\varphi} \in FC\ (\mathbf{A}/\theta)$, then $\varphi \in FC(\mathbf{A})$.

The following theorem shows that the algebra $\mathbf{A}/\theta$ is isomorphic to $\mathbf{A}/\theta^*$ in a natural way.

Theorem 5.13 If $\theta \in FC(\mathbf{A})$, then $\mathbf{A}/\theta \cong \mathbf{A}/\theta^*$.

Proof. Let $\alpha : \mathbf{A}/\theta \to \mathbf{A}/\theta^*$ be defined by $\alpha\,(a\theta) = a\theta^*$ for all $a \in A$.

(i) $a\theta = b\theta$ iff for all x, $\theta\,(a,x) = \theta\,(b,x)$; thus $\theta\,(a,b) = \theta\,(b,b) = 1$ which implies that $a\theta^*b$. Hence $a\theta^* = b\theta^*$. This shows that α is well-defined.

(ii) For any n -ary function $f^* \in F^*$ define an n-ary function $(f^*)'$ an $\mathbf{A}/\theta^*$ by $(f^*)'(a_1\theta^*,\ldots,a_n\theta^*) = f(a_1,\ldots,a_n)\,\theta^*$, then $(f^*)'$is in the set of aperations on $\mathbf{A}/\theta^*$ by the definition of an ordinary quotient algebra and we have

$$\alpha\,(f^*(a_1\theta,\ a_2\theta,\ldots,a_n\theta)) = \alpha\,(f(a_1,\ a_2,\ldots,\ a_n)\ \theta)$$

$$= f(a_1,\ a_2,\ldots,\ a_n)\ \theta^*$$

$$=(f^*)'(a_1\theta^*,\ a_2\theta^*,\ldots,\ a_n\theta^*)$$

$$=(f^*)'(\alpha\ (a_1\theta),\ \alpha\ (a_2\theta),\ldots,\ \alpha\ (a_n\theta))$$

for any $a_1, a_2,\ldots,a_n \in A$. and for any $c \in C$, we have $\alpha\,(c\theta) = c\theta^*$. So α is a homomorphism.

(iii) If $a, b \in A$ and $\alpha\,(a\theta) = \alpha\,(b\theta)$, then

$$a\theta^* = b\theta^* \Rightarrow a\theta^*\ b \Rightarrow \theta\ (a,b) = 1.$$

Now, for any $c \in A$

$$a\,\theta\,(c) = \theta\,(a,c) = \theta\,(a,c) \wedge \theta\,(a,b) \le \theta\,(b,c) = b\theta(c).$$

Similarly, $b\theta(c) \le a\theta(c)$. Hence $a\theta = b\theta$ and α is one-to-one.

(iii) If $a\theta^* \in A/\theta^*$, then $a\theta \in A/\theta$ is such that $\alpha\,(a\theta) = a\theta^*$, which makes α onto.

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(Received 22/04/1992;
in revised form 29/09/1992)

بعض البنى الجبرية المشوشة للجبور الشاملة

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ص. ب: ٢٤٥٥ - الرياض ١١٤٥١ - المملكة العربية السعودية

لقد بدأ إهتمام الرياضيين في نظرية المجموعات المشوشة عام ١٩٦٥ م عندما نشر زادهِ (Zadeh) بحثه المشهور «المجموعات المشوشة». وأول من وظف هذا المفهوم لدراسة البنى الجبرية العالم الرياضي روزنفلد (Rosenfeld) عام ١٩٧١ م حيث درس مفهوم الزمر المشوشة. من بعد ذلك كثر الاهتمام بمفهوم البنى الجبرية المشوشة حيث تم نشر الكثير من البحوث في السنوات العشرين الماضية في بعض مفاهيم الزمر المشوشة، الحلقات والحقول المشوشة، الفضاءات الخطية المشوشة، والشبكات المشوشة. لقد تم برهان كثير من النتائج التي أمكن تعميمها في مضمار المجموعات المشوشة ولكن تبين لسوء الحظ أن هناك مفاهيم كثيرة لا يمكن تعميمها.

لقد قدم العالمان جيرلا وتورتورا (Gerla and Tortora) مفهوم الجبر الجزئي المشوش من الجبر الشامل. في هذا البحث نواصل دراسة مفهوم الجبر الجزئي المشوش حيث نبرهن أن الصورة والصورة العكسية لجبر جزئي مشوش تحت تأثير التشاكل هو جبر جزئي مشوش. كما نقدم في بحثنا فكرة الجبر الجزئي المشوش المولد من قبل مجموعة مشوشة ونعطي طريقة لبناء الجبر الجزئي المشوش من مجموعة جزئية مشوشة ذات مدى منتهي. كذلك نبرهن بعض النظريات التشخيصية.

لقد تبين لنا أن مجموعة جميع الجبور الجزئية المشوشة من جبر معطى تكون شبكية. كذلك نقدم هنا فكرة التطابقات المشوشة، صفوف التطابقات المشوشة والجبور الخارجة المشوشة ونبرهن بأن الجبور الخارجة المشوشة تماثل بطريقة طبيعية خارجة اعتيادية معينة.

خلال هذا البحث سنرمز للفترة المغلقة $[0,1]$ بالرمز I. ونعرف المجموعة الجزئية المشوشة من مجوعة غير خالية X بأنها التطبيق $\mu: X \rightarrow I$، والعلاقات المشوشة على X بأنها التطبيق $\theta: X \times X \rightarrow I$. إذا كان $a,b \in I$ فإن $a \wedge b$ تعني $\min(a,b)$ و $a \vee b$ تعني $\max(a,b)$. إذا كانت υ,μ مجموعتين مشوشتين جزئيتين من المجموعة X فإن $\mu \leqslant \upsilon$ تعني $\mu(x) \leqslant \upsilon(x)$ لكل $x \in X$. تسمى العلاقة المشوشة θ على X بأنها علاقة تكافؤ مشوشة على X إذا كان $\theta(x,x) = 1$، $\theta(x,y) = \theta(y,x)$ و $\theta(x,y) \geqslant \theta(x,z) \wedge \theta(z,y)$ لكل $x,y,z \in X$. سنرمز كذلك للتطبيق المميز للمجموعة X بالرمز 1_X.