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On Almost Unitarily Equivalent Operators

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ABSTRACT. The almost unitarily equivalence relation, $\ominus$, between operators is defined. It is shown that some properties are shared by almost unitarily equivalent operators. Various results related to $\ominus$ are proved.

1 Let H be a Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H. If S and T are in $L(H)$, then S and T are called unitarily equivalent if there is a unitary operator U such that $S = U^{-1}TU$, or equivalently, $S = U^*TU$. We call S and T almost unitarily equivalent, $S \ominus T$, if there is a unitary operator U such that the following two conditions are satisfied:

$$T^*T = U^*S^*SU, \tag{a}$$

$$T^* + T = U^* (S^* + S) U. \tag{b}$$

If $S,T,F \in L(H)$, then it can be easily shown that (i) $S \ominus S$, (ii) $S \ominus T$ if and only if $T \ominus S$, and (iii) if $S \ominus T$, and $T \ominus F$, then $S \ominus F$.

In the first section of this paper we show that some properties are shared by almost unitarily equivalent operators.

Proposition 1.1 *If* $T \in L(H)$ such that $T \ominus 0$, then $T = 0$.

Proof. $T \ominus 0$ implies that there exists a unitary operator U such that $0 = U^*T^*TU$ which implies that $T^*T = 0$. Thus, by (Berberian 1976, Theorem 2(6) $T = 0$.

Proposition 1.2 *If* $T \in L(H)$ such that $T \ominus 1$, then $T = I$.

Proof. Let $A + iB$ be the cartesian decomposition of T, then $T \ominus I$ implies that there is a unitary operator U such that

$$I^* I = U^* (T^*T) U, \tag{i}$$

$$I^* + I = U^* (T^* + T) U, \tag{ii}$$

From (ii) we conclude that $2I = U^*(2A)U$ which implies that $A = UU^* = I$. From (i) we conclude that $I = T^*T = (A^2 + B^2) + i(AB - BA)$. Thus $I = A^2 + B^2$ which implies (since $A^2 = I$) that $B^2 = 0$. Since B is hermitian, $B = 0$. Thus $T = A = I$.

Definition 1.1 The numerical range, $W(T)$, of an operator T in $L(H)$ is the set of all complex numbers of the form $\langle Tf,f \rangle$, where f varies over all vectors on the unit sphere. The numerical radius, $w(T)$, of T is defined by $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$.

Proposition 1.3 *If* $S, T \in L(H)$ such that $S \ominus T$, then $w(T^*T) = w(S^*S)$.

Proof. By assumption there is a unitary operator U such that $T^*T = U^* S^* SU$. Hence, $w(T^*T) = w(U^*S^*SU))$. Since w is unitarily invariant in the sense that $w(U^*TU) = w(T)$ for any operator T and any unitary U (Halmos 1982) we have $w(T^*T) = w(S^*S)$.

Corollary 1.1 *If* $S,T \in L(H)$ such that $S \ominus T$, then $\|S\| = \|T\|$.

Proof. Since $S \ominus T$, $T^*T = U^*S^*SU$ which implies that $\|T^*T\| = \|S^*S\|$. Since $\|A^*A\| = \|A\|^2$ for any $A \in L(H)$, $\|T\|^2 = \|S\|^2$ which implies that $\|S\| = \|T\|$.

Corollary 1.2 *If* $S,T \in L(H)$ such that $S \ominus T$ and S is a contraction, then T is a contraction.

Proof. Since $S \ominus T$, $\|S\| = \|T\|$. Since $\|S\| \leq 1$, $\|T\|. \leq 1$, that is T is a contraction.

Proposition 1.4 *If* $S, T \in L(H)$ such that $S \ominus T$ and T is partially isometric, then S is partially isometric.

Proof. Since $S \ominus T$, there is a unitary operator U such that $U^*S^*SU = T^*T$. Since T is partially isometric, T^*T is a projection. Thus U^*S^*SU is a projection which means that $(U^*S^*SU)(U^*S^*SU) = U^*S^*SU$. This implies that $U^*S^*SS^*SU = U^*S^*SU$, which implies that $UU^*S^*SS^*SUU^* = UU^*S^*SUU^*$. Thus, we have $(S^*S)^2 = S^*S$, which means that S^*S is a projection. Thus S is partially isometric.

Definition 1.2 $T \in L(H)$ is called a θ-operator if $T^* + T$ commutes with T^*T. The class of all θ-operators in $L(H)$ is denoted by θ.

Proposition 1.5 If S, T are in *L(H) such that $T \in \theta$ and $T \ominus S$, then $S \in \theta$.*

Proof. $T \ominus S$ implies that there exists a unitary operator U such that $U^*T^*TU = S^*S$ and $U^*(T^* + T)U = S^* + S$. Thus we have $(U^*T^*TU)\,[U^*(T^* + T)U] = S^*S(S^* + S)$ which implies that

$$U^*T^*T(T^* + T)U = S^*S(S^* + S). \tag{A}$$

Also, we have $[U^*(T^* + T)U]\,(U^*T^*TU) = (S^* + S)S^*S$ which implies that

$$U^*(T^* + T)T^*TU = (S^* + S)S^*S. \tag{B}$$

Since $T \in \theta$, then the left hand sides of (A) and (B) are equal, which implies that $(S^* + S)S^*S = S^*S(S^* + S)$. Thus $S \in \theta$.

Proposition 1.6 If S, $T \in L(H)$ such that $S \ominus T$ and S is compact, then T is compact.

Proof. Since $S \ominus T$, there is a unitary operator U such that $T^*T = U^*S^*SU$. Since S is compact, U^*S^*SU is compact which implies that T^*T is compact. Thus, by (Kreyszig 1978), T is compact.

Definition 1.3 An operator $T \in L(H)$ is called skew-adjoint in case $T^* = -T$.

Proposition 1.7 If S, $T \in L(H)$ such that $S \ominus T$ and S is skew-adjoint, then T is skew-adjoint.

Proof. Since $S \ominus T$, there is a unitary operator U such that $U^*(T^*+T)U = S^*+S = 0$ (since S is skew-adjoint). Thus, $T^* + T = 0$ which means that T is skew-adjoint.

Before we give the next proposition we need the following theorem.

Theorem 1.1 An operator $T \in L(H)$ is hermitian if and only if $(T + T^*)^2 \geq 4\, T^*T$.

Proof (Istratescu 1981).

Proposition 1.8 If $S,T \in L(H)$ such that $S \ominus T$ and S is hermitian, then T is hermitian.

Proof. Since $S \ominus T$, there is a unitary operator U such that $U^*S^*SU = T^*T$, which implies that

$$U^*(4S^*S)U = 4T^*T. \tag{i}$$

Also, $S \ominus T$ implies that $U^*(S^*+S)U = T^* + T$, which implies that $U^*(S^* + S)\ UU^*(S^* + S)\ U\ +\ (T^*+ T)^2$. Thus

$$U^*\ (S^* + S)^2 U = (T^* + T)^2. \tag{ii}$$

Since S is hermitian, $(S^* + S)^2 = 4S^*S$. Substituting in (ii) we get $U^*\ (4S^*S)\ U = (T^* + T)^2$, which implies, by (i), that $(T^* + T)^2 = 4T^*T$. Thus, by the above theorem, T is hermitian.

Proposition 1.9 If $S,T \in L(H)$ such that $S \ominus T$ and S is a projection, then T is a projection.

Proof. $S \ominus T$ implies that there is a unitary operator U such that $S^*S = U^*T^*TU$ and $S^* + S = U^*(T^* + T)\ U$. Since S is a projection, it is hermitian and $S^2 = S$. By proposition 1.9, T is hermitian. Thus $S = U^*T^2U$ and $2S = U^*(2T)\ U$ which implies that $T^2 = T$. Hence T is a projection.

2 In this section we prove various results related to $\ominus$.

Proposition 2.1 If $S,T \in L(H)$ are unitarily equivalent, then $S \ominus T$.

Proof. Since S and T are unitarily equivalent, then there is a unitary operator U such that $T = U^*SU$. Thus $T^* = U^*S^*U$, which implies that $T^*T = U^*S^*UU^*SU = U^*S^*SU$, and $T^* + T = U^*S^*U + U^*SU = U^*(S^* + S)\, U$. Thus $S \ominus T$.

Proposition 2.2 If $S,T \in L(H)$ such that $S \ominus T$ and S is hermitian, then S and T are unitarily equivalent.

Proof. $S \ominus T$, implies that there is a unitary operator U such that $T^* + T = U^*(S^* + S)\, U$. Since S is hermitian, by Proposition 1.9, T is also hermitian. Thus, we have $2T = U^*(2S)U$ which implies that $T = U^*SU$. Hence T and S are unitarily equivalent.

Proposition 2.3 if $T \in L(H)$ is in θ, then there is a normal operator N in L(H) with $T \ominus N$.

Proof. Consider the operator

$$N = \frac{T^* + T + i\sqrt{4T^*T - (T^* + T)^2}}{2};$$

then by (Campbell and Gellar 1977, p.305) N is normal with $N^*N = T^*T$ and $N^* + N$

$= T^* + T$. Since the identity operator I is unitary and since $T^*T = I^*N^*NI$ and

$T^* + T = I^*(N^* + N)I$, $T \ominus N$.

Lemma 2.1 If $S,T \in L(H)$ such that T is isometric and $S \ominus T$, then S is isometric.

Proof. $S \ominus T$ implies that there is a unitary operator U such that $T^*T = U^*S^*SU$. Since T is isometric, $T^*T = I$ which implies that $S^*S = I$. Thus S is isometric.

Next, we give a characterization of isometric operators in terms of $\ominus$.

Proposition 2.4 $T \in L(H)$ is isometric if and only if $T \ominus U$ for some unitary U.

Proof. Suppose that T is isometric; then it is in θ. Thus, by Proposition 2.1 there is a normal operator N with $T \ominus N$. By Lemma 2.1, N is isometric. Thus N is unitary.

Now, suppose that $T \ominus U$ for some unitary U; then there is a unitary V with $V^*T^*TV = U^*U = I$. This implies that $T^*T = VV^* = I$. Thus T is isometric.

Let $T \in L(H)$ be unitary. Then $T^*T = TT^*$ $(=I)$. Thus $T^*T = I^*TT^*I$ which means that $T \ominus T^*$. Since $T^* = T^{-1}$, we have $T \ominus T^{-1}$. If $T \in L(H)$ such that $T \ominus T^{-1}$, then it is not necessary that T is unitary, as the following example shows.

Example: Consider the operator $T = \begin{vmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{vmatrix}$ on the two-dimensional space R^2.

Then it can be shown that $T^2 = I$ which implies that $T = T^{-1}$. Thus $T \ominus T^{-1}$. However, $\|T\| > 1$ which means that T is not unitary.

In the next proposition, we give a condition under which $T \ominus T^{-1}$ implies that T is a scalar multiple of a unitary operator. First we need the following result.

Theorem 2.2 An invertible operator $T \in L(H)$ is a scalar multiple of a unitary operator if and only if $\|T\| \, \|T^{-1}\| = 1$.

Proof. (Shah and Sheth 1975, p.181).

Proposition 2.5 Let $T \in L(H)$ such that $T \ominus T^{-1}$ and T is a contraction; then T is a scalar multiple of a unitary operator.

Proof. Since T is a contraction and $T \ominus T^{-1}$, then, by Corollary 1.2, T^{-1} is a contraction. Thus $\|T^{-1}\| \leq 1$ which implies that $\|T\| \, \|T^{-1}\| \leq 1$. On the other hand we have $\|T\| \, \|T^{-1}\| \geq \|TT^{-1}\| = \|I\| = 1$. Thus $\|T\| \, \|T^{-1}\| = 1$ which implies, by the above theorem, that T is a scalar multiple of a unitary operator.

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المؤثرات المتكافئة أحادياً تقريباً

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نعرف في هذا البحث علاقة «متكافئة أحادياً تقريباً» ـ $\ominus$ ـ بين المؤثرات على فضاء هلبرت H على النحو التالي: إذا كان S، T مؤثرين في $L(H)$ ـ مجموعة المؤثرات الخطية المحدودة المؤثرة على H ـ فإن S، T يكونان متكافئين أحادياً تقريباً، وبالرموز $S \ominus T$، إذا كان هناك مؤثراً أحادياً، U، في $L(H)$ بحيث يتحقق الشرطان: $T^*T = U^*S^*SU$ و $T^* + T = U^*(S^* + S)U$.

في الفصل الأول من هذا البحث نثبت أنه هناك مجموعة من الخصائص المشتركة بين المؤثرات المتكافئة أحادياً تقريباً. فنثبت مثلاً أنه إذا كان S,T مؤثرين في $L(H)$ بحيث $S \ominus T$ وكان T هو المؤثر الصفري (وعلى الترتيب المؤثر المحايد، مؤثراً متشاكلاً جزئياً، مؤثراً منكمشاً، مؤثراً من النوع $\ominus$، مؤثراً متشاكلاً، مؤثراً متراصاً، مؤثراً متجاوراً ملتفاً، مؤثراً ذاتي التجاور، مؤثراً إسقاطياً) فإن المؤثر S يكون له نفس الخاصية. كذلك نثبت أنه إذا كان S، T مؤثرين في $L(H)$ وكان $S \ominus T$ فان أنصاف الاقطار العددية للمؤثرين S ، T يكونان متساويين، ومن هذه النتيجة الأخيرة نستنتج أنه إذا كان $S \ominus T$ فإن $\|T\| = \|S\|$. في الفصل الثاني من هذا البحث نثبت مجموعة من النتائج العامة المتعلقة بالعلاقة $\ominus$. فمثلاً نثبت أنه إذا كان S، T مؤثرين في $L(H)$ ومتكافئين أحادياً فإنهما متكافئين أحادياً تقريباً. كذلك نثبت أنه إذا كان T مؤثراً في $L(H)$ ومن النوع $\ominus$ فإنه يوجد في $L(H)$ مؤثراً عمودياً، N، بحيث يكون

$N \ominus T$. ونستفيد من هذه النتيجة الأخيرة في إعطاء تمييز للمؤثرات المتشاكلة بدلالة العلاقة $\ominus$ حيث نثبت أن المؤثر T في $L\,(H)$ يكون متشاكلاً إذا وفقط إذا وجد في $L\,(H)$ مؤثراً أحادياً، U ، بحيث يكون $U \ominus T$. وفي نهاية هذا البحث نعطي مثلاً على مؤثر في $L\,(H)$ بحيث يكون $T \ominus T^{-1}$ ولكن T غير أحادي .

ثم نختتم البحث بإثبات أنه إذا كان T مؤثراً في $L\,(H)$ وكان $T \ominus T^{-1}$ فإن T يكون مؤثراً أحادياً بشرط أن يكون مؤثراً منكمشاً .