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A Global Version of Bôcher's Theorem

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ABSTRACT. A necessary condition for a subharmonic function in $\mathbb{R}^n$ to have a non-constant harmonic minorant is obtained. This result is shown to be a variation of Bôcher's theorem in $\mathbb{R}^n$.

1. Preliminaries

Let E_n denote the fundamental singularity of the Laplacian operator Δ at $0 \in \mathbb{R}^n$, $n \geq 2$. It is given by $\frac{1}{2\pi}\log|x|$ when $n = 2$ and $-1/(n-2)\,\sigma_n\,|x|^{n-2}$ when $n \geq 3$, σ_n being the area of the unit sphere in $\mathbb{R}^n$. For any compact set K in $\mathbb{R}^n$, $H_0\,(\mathbb{R}^n \setminus K)$ will denote the set of harmonic functions in $\mathbb{R}^n \setminus K$ which behave like $E_n\,(x)$ as $|x| \to \infty$. In other words, $s \in H_0\,(\mathbb{R}^n \setminus K)$ if there is a constant α such that, as $|x| \to \infty$, $|s(x) - \alpha \log|x|| \to 0$ if $n = 2$, and $|s(x)| \leq |\alpha| / |x|^{n-2}$ if $n \geq 3$ (see Anandam and Al-Gwaiz (1993)).

We first note that any function in $H_0(\mathbb{R}^n)$ must be the constant 0. This follows from

Lemma 1.1

If a function u is harmonic in $\mathbb{R}^n$ and majorized, i.e. bounded above, by a function in H_0 outside a compact set, then u is a constant.

Proof.

If $n \geq 3$, any function in H_0 tends to 0 as $|x| \to \infty$, hence $\limsup_{|x| \to \infty} u(x) \leq 0$ and, by the maximum principle, u is a constant in $\mathbb{R}^n$. In $\mathbb{R}^2$ we first use the mean value property to convert the growth condition on u to a condition on $|u|$, and then the divergence theorem to show that the partial derivatives $\partial_1 u = \partial u/\partial x_1$ and $\partial_2 u = \partial u/\partial x_2$ are both 0.

If u is bounded above by a function in H_0 outside a compact set in $\mathbb{R}^2$, then there are numbers $R > 1$ and $\alpha > 0$ such that $u(x) \leq \alpha \log |x|$ for all $|x| \geq R$. Since u is harmonic in $\mathbb{R}^2$, its positive part $u^+ = \frac{1}{2}(|u| + u)$ is a subharmonic function which satisfies

$$u^+(x) \leq \alpha \log |x| \text{ for all } |x| \geq R.$$

Hence, if $M(r,u^+)$ denotes the mean value of u^+ on $|x| = r$, then

$$M(r,u^+) \leq \alpha \log r \text{ for all } r \geq R.$$

Since $|u| = 2u^+ - u$, this implies that

$$M(r,|u|) \leq 2\alpha \log r - u(0) \tag{1.1}$$

and consequently $\frac{1}{r} M(r, |u|) \to 0$ as $r \to \infty$.

Let $x_0 \in \mathbb{R}^2$ satisfy $|x_0| = 2R$ and $D = \{x \in \mathbb{R}^2 : |x - x_0| < R\}$. Since $\partial_1 u$ is harmonic in $\mathbb{R}^2$, we have by the mean value property

$$|\partial_1 u(x_0)| = \frac{1}{\pi R^2} \int_D \partial_1 u(x)dx.$$

Now the divergence theorem gives

$$|\partial_1 u(x_0)| = \frac{1}{\pi R^2} \left| \int_0^{2\pi} u(R,\theta) \cos\theta \, Rd\theta \right| \leq \frac{2}{R} M(R,|u|).$$

As $|x_0| = 2R \to \infty$ we conclude, in view of (1.1), that $\partial_1 u(x_0) \to 0$. Consequently $\partial_1 u \equiv 0$. Similarly $\partial_2 u \equiv 0$, and hence u is constant in $\mathbb{R}^2$. □

Bôcher's theorem characterizes the behaviour of a positive harmonic function in the neighbourhood of an isolated singularity $x_0 \in \mathbb{R}^n$. There is no loss of generality in choosing the point x_0 to be the origin 0, and the neighbourhood to be the unit ball $B_n = \{x \in \mathbb{R}^n : |x| < 1\}$. Bôcher's theorem then states (see Helms 1969): If u is a non-negative harmonic function in $B_n \setminus \{0\}$, there is a constant $\alpha \le 0$ and a harmonic function v in B_n such that $u = \alpha E_n + v$ in $B_n \setminus \{0\}$.

Noting that the inversion $x \mapsto x/|x|^2$ in $\mathbb{R}^n \setminus \{0\}$ preserves positivity and harmonicity, the function

$$u^*(x) = \frac{1}{|x|^{n-2}} \, u\left(\frac{x}{|x|^2}\right), \quad |x| > 1,$$

is positive and harmonic in $|x| > 1$ if and only if u is positive and harmonic in $0 < |x| < 1$. Thus, if u satisfies the hypothesis of Bôcher's theorem, then

$$u^*(x) = \frac{1}{|x|^{n-2}} \left[\alpha E_n\left(\frac{x}{|x|^2}\right) + v\left(\frac{x}{|x|^2}\right)\right] \text{ in } |x| > 1. \tag{1.2}$$

As $|x| \to \infty$, the right-hand side of (1.2) is of the form $-\frac{\alpha}{2\pi}\log|x| + v(0) + o(1)$ if $n = 2$, and of the form $-\frac{\alpha}{(n-2)\sigma_n} + v(0)|x|^{2-n} + o(|x|^{2-n})$ if $n \ge 3$. Consequently Bôcher's theorem has an equivalent version where the singular point is at ∞:

Bôcher's theorem

Let u be a non-negative harmonic function in $B^\star_n = \{x \in \mathbb{R}^n : | x | > 1\}$. Then u = s + c, where $s \in H_0 (B^\star_n)$ and c is a constant.

Corollary

Let u be a harmonic function which is defined outside a compact set K in $\mathbb{R}^n$, $n \geq 3$, and bounded on one side. Then u (x) has a finite limit as $| x | \to \infty$. Moreover, $u(x) \to 0$ if and only if $u \in H_0 (\mathbb{R}^n \setminus K)$.

2. L-Potentials

Definition 2.1.

A subharmonic function in $\mathbb{R}^n$ is admissible if it has a harmonic majorant outside a compact set.

Proposition 2.1.

If s is an admissible subharmonic function in $\mathbb{R}^n$, then s has a harmonic majorant in $B^\star_n$.

Proof.

s being admissible, there is an $R > 1$ and a harmonic function u in $| x | \geq R$ which majorizes s in $| x | > R$. For any $a > R$, let

$$h(x) = \begin{cases} \log \dfrac{| x |}{a} & \text{in } \mathbb{R}^2 \\ \dfrac{1}{a^{n-2}} - \dfrac{1}{| x |^{n-2}} & \text{in } \mathbb{R}^n,\ n \geq 3, \end{cases}$$

and let v be a harmonic function in $1 < | x | < a$ such that

$$v(x) = \begin{cases} 0 \text{ on } | x | = 1 \\ u(x) \text{ on } | x | = a. \end{cases}$$

Now consider the function $\alpha h + v$ where α is a positive constant. Since $h(x) < 0$ in $| x | < a$, we can make α large enough so that $\alpha h(x) + v(x) < u(x)$ on $| x | = R$. By the maximum principle, $\alpha h(x) + v(x) \leq u(x)$ on $R < | x | < a$. Define

$$t(x) = \begin{cases} u(x) \text{ in } |x| > a \\ \alpha h(x) + v(x) \text{ in } 1 < |x| \le a, \end{cases}$$

so that t is a continuous superharmonic function in $|x| > 1$. Since the subharmonic function s is upper bounded in $1 \le |x| \le a$, there is a positive number c such that $t(x) + c \ge s(x)$ in $1 < |x| \le a$, and hence $t(x) + c \ge s(x)$ in $|x| > 1$. Consequently there is a harmonic function h in $|x| > 1$ such that $t(x) + c \ge h(x) \ge s(x)$ in $|x| > 1$. □

Definition 2.2

An admissible subharmonic function s in $\mathbb{R}^n$ is an L-potential if the least harmonic majorant (*LHM*) h of s in $B^\star_n$ *is in* $H_0(B^\star_n)$.

Proposition 2.2

In $\mathbb{R}^n$, $n \ge 3$, s is an L-potential if and only if $-s$ is a Newtonian potential, in the sense that $s = E_n * \mu$ for some positive measure μ in $\mathbb{R}^n$.

Proof.

If s is an L-Potential in $\mathbb{R}^n$, $n \ge 3$, then $s \le h$ in $B^\star_n$ for some $h \in H_0(B^\star_n)$. Hence h satisfies the condition $|h(x)| \le |\alpha| / |x|^{n-2}$ in a neighbourhood of the point at ∞. Since $h(x) \to 0$ as $|x| \to \infty$, the maximum principle implies that $s \le 0$ in $\mathbb{R}^n$. Furthermore, $\limsup_{|x| \to \infty} s(x) = 0$ because h is the *LHM* of s in $B^\star_n$. By the Riesz representation, $-$ s is therefore a Newtonian potential in $\mathbb{R}^n$. □

Proposition 2.3

Every admissible subharmonic function in $\mathbb{R}^n$, $n \ge 2$, is the unique sum of an L-potential and a harmonic function in $\mathbb{R}^n$.

Proof.

Let s be an admissible subharmonic function in $\mathbb{R}^n$, and u be the *LHM* of s is $B^\star_n$. Then u is of the form $u = v + h$, where $v \in H_0(B^\star_n)$ and h is harmonic in $\mathbb{R}^n$ (see Anandam and Al-Gwaiz 1993, theorem 2.2). Consequently, if we set $q = s - h$, then q is subharmonic in $\mathbb{R}^n$ and v is its *LHM* in $B^\star_n$. Since $v \in H_0(B^\star_n)$, q is an L-potential, and $s = q + h$ is the required decomposition.

To show that this decomposition is unique, suppose $s = q' + h'$ is another such decomposition. Then $q = q' + h' - h$ in $\mathbb{R}^n$. Now if v and v' are the *LHM* of q and q', respectively, in $B^\star_n$, then the *LHM* of $q = q' + h' - h$ is $v' + h' - h$. Hence $v = v' + h' - h$, so that $h' - h = v - v' \in H_0$. By the remark preceding lemma 1.1, $h' - h \equiv 0$. □

Remark: From propositions 2.2 and 2.3, it follows that if s is a subharmonic function in $\mathbb{R}^n$, $n \geq 3$, with the associated measure $d\mu(x) = \Delta sdx / (n-2)\ \sigma_n$ (in the sense of distributions), then s is admissible if and only if

$$\int_{|x|>1} \frac{d\mu(x)}{|x|^{n-2}} \text{ is finite.}$$

When $n = 2$, the same result is true with $d\mu(x) = \Delta sdx/2\pi$. To see this, note that a subharmonic function s in $\mathbb{R}^2$ has a harmonic majorant outside a compact set if and only if the flux of s at infinity is finite. But the flux of s is a constant multiple of the total measure associated with s.

3. Global Version of Bôcher's Theorem

Theorem 3.1

Every admissible subharmonic function $\mathbb{R}^n$, $n \geq 2$, which is bounded below, is an L-potential up to an additive constant.

Proof.

Let s be an admissible subharmonic function in $\mathbb{R}^n$ such that $s \geq m$, where m is a constant. By proposition 2.3, $s = q + h$, where q is an L-potential and h is a harmonic function in $\mathbb{R}^n$. Let $v \in H_0$ be the *LHM* of q in $B^\star_n$. Since $q + h \geq m$, $v \geq m - h$ in $B^\star_n$. By lemma 1.1, $m - h$, and hence h, is a constant. □

This theorem provides a necessary condition for a subharmonic function in $\mathbb{R}^n$ to have a non-constant harmonic minorant:

Corollary

If s is a subharmonic function minorized by a non-constant harmonic function in $\mathbb{R}^n$, then $s^+ = \frac{1}{2}(|s| + s)$ is not admissible.

Proof.

Suppose u is a non-constant harmonic function in $\mathbb{R}^n$ such that $u \leq s$. If s^+ were admissible then, by theorem 3.1, it would be an L-potential up to an additive constant. By lemma 1.1, u is then a constant, thereby contradicting the hypothesis. □

We shall now prove that theorem 3.1, which expresses a global property of admissible subharmonic functions in $\mathbb{R}^n$, is a variant of Bôcher's theorem, which expresses a local property of positive harmonic functions in a punctured neighbourhood of a point in $\mathbb{R}^n \cup \{\infty\}$. Here we shall use the version of Bôcher's theorem where the singular point is taken at infinity:

(i) Let $u \geq 0$ be a harmonic function in $|x| \geq a$ with $0 < a < 1$. Define

$$h_n(x) = \begin{cases} \log \frac{|x|}{a} \text{ if } n = 2 \\ 1 - \frac{a^{n-2}}{|x|^{n-2}} \text{ if } n \geq 3. \end{cases}$$

Then $h_n(x)$ is positive harmonic in $|x| > a$ and tends to 0 as $|x| \to a$.

For any continuous function f on the unit sphere $|x| = 1$, let $D_1 f$ denote the Dirichlet solution in $|x| < 1$ with boundary value f on $|x| = 1$, i.e. $D_1 f$ is the harmonic function in $|x| < 1$ which tends to f as $|x| \to 1$. If $\alpha > 0$ is chosen large enough, then

$$D_1(u + \alpha h_n) \geq u = u + \alpha h_n \text{ on } |x| = a.$$

This implies that $D_1(u + \alpha h_n) \geq u + \alpha h_n$ in $a \leq |x| \leq 1$. Hence

$$s = \begin{cases} u + \alpha h_n \text{ in } |x| > 1 \\ D_1(u + \alpha h_n) \text{ in } |x| \leq 1 \end{cases}$$

is a continuous admissible subharmonic function in $\mathbb{R}^n$ which is positive, since $u \geq 0$ by hypothesis and both α and h_n are positive. By theorem 3.1, s is therefore an L-potential up to an additive constant. Hence the *LHM* of s in $|x| > 1$ is of the form v + a constant, where $v \in H_0$.

But since s is harmonic in $|x| > 1$, its *LHM* in $|x| > 1$ is s itself. Therefore $u + \alpha h_n = v$ + a constant, with $v \in H_0$. Now h_n, up to an additive constant, belongs to H_0, hence the same is true of u. This proves Bôcher's theorem.

(ii) On the other hand, let $s \geq m$ be an admissible subharmonic function in $\mathbb{R}^n$, and suppose h is the *LHM* of s in $|x| > 1$. Since $h \geq m$, Bôcher's theorem implies that, up to an additive constant, $h \in H_0$. Hence, up to an additive constant, s is an L-potential.

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