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Four Dimensional CR-Submanifolds of the Six-Dimensional Sphere

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ABSTRACT . We consider 4-dimensional CR-Submanifolds of constant curvature in the six-dimensional sphere. We proved that such submanifolds does not exist if the holomorphic distribution is parallel. We also proved that the integral submanifold of the holomorphic distribution is totally geodesic.

Introduction: An almost Hermitian manifold $(\overline{M},J,g)$ with Riemannian connection $\overline{\nabla}$ is called nearly Kaehlerian if $(\overline{\nabla}_X J)(X) = 0$ for any $X \varepsilon \mathfrak{X}(M)$. The typical important example is the six-dimentional sphere S^6 (Fukami and Ishihara 1955). It is because of this nearly Kaehler, non-Kaehler structure that S^6 has drawn the attention. Different classes of submanifolds of S^6 have been studied by several authors. In particular (Sekigawa 1984) has proved that S^6 does not admit any proper CR-product. (Gray 1969) has also proved that S^6 does not admit a 4-dimensional complex submanifolds. It is obvious that a 4-dimensional totally real submenifold in S^6 does not exits. Therefore the only four-dimensional submanifolds in S^6, if they exist, are the proper CR-submanifolds.

In this paper we consider 4-dimensional CR-submanifolds M of S^6, with parallel holomorphic distribution D, we prove that S^6 does not admit such submanifolds with constant curvature. We also prove that the integral submanifold of the distribution D is totally geodesic .

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Preliminaries: A submanifold M of dim (2p+q) in S^6 is called a CR-submanifold if there exists a pair of orthogonal complementary distributions D and $D^\perp$ auch that JD=D and $JD^\perp \subset \nu$ where ν is the normal bundle of M and dim D = 2p, dim $D^\perp$ = q (Bejancu 1978). Thus the normal bundle ν splits as $\nu = JD^\perp \oplus \mu$ where μ is invariant sub-bundle of ν under J. A CR-submanifold is said to be proper if neither D = {0 }, nor $D^\perp$ = { 0 }.

Let ∇ be the Riemannian connection induced on M and $\nabla^\perp$ be that in the normal bundle. Then we have the following Gauss and Weingarten formulas.

$$(1.1)\quad \overline{\nabla}_X Y = \nabla_X Y + h\,(X, Y) \qquad X, Y \in \mathfrak{X}(M)$$

$$(1.2)\quad \overline{\nabla}_X N = -A_N X + \nabla^\perp_X N\ ,\ N \in \nu.$$

where h (X, Y) and $A_N X$ are the second fundamental forms which are related by

$$g\,(h(X,Y)\,,N) = g\,(A_N X,\, Y)$$

A CR-submanifold is said to be totally geodesic if h = 0 and mixed totally geodesic if h (X, Z) = 0 for $X \in D$ and $Z \in D^\perp$.

The holomorphic distribution D is said to be parallel if $\nabla_Y X \in D$ for any X, Y in D.

Let R be the curvature tensor associated with ∇. Then the equation of Gauss is given by

$$(1.3)\quad R(X,Y,Z,W) = \overline{R}\,(X,Y,Z,W) + g\,(h(X,W),\, h\,(Y,Z)) - g\,(h\,(X,Z),\, h\,(Y,W)).$$

where in this case R (X,Y,Z,W) = $\overline{C}$ {g (Y,Z) g(X,W) -g(X,Z) g(Y,W) } is the Riemannian curvature of S^6 with constant curvature C. The CR-submanifold M is a space of constant curvature if all sectional curvatures $R(X,Y,Y,X)$ are constants for any plane section $\{X,Y\}$, where $X,Y \in \mathfrak{X}(M)$.

4- *Dimensional CR-submanifolds of* S^6: First we prove the following lemma

Lemma: In a 4-dimensional CR-submanifold of S^6 with parallel holomorphic distribution D we have $h(X,Y) = 0$ for X,Y in D.

Proof:

We know that in S^6 there does not exist a 4-dimensional holomorphic submanifold (Gray 1969). Also S^6 does not admit 4-dimensional totally real submanifolds. Therefore we have only proper CR-submanifold of dimension 4 in S^6. *i.e.* $\dim D = \dim D^{\perp} = 2$. Hence the normal bundle $\nu \equiv JD^{\perp}$. Therefore $h(X,Y) \in JD^{\perp}$. Also for X non-zero in D, $\{X, JX\}$ is a local frame for D, and therefore it follows from $(\overline{\nabla}_X J)X = 0$, that $(\overline{\nabla}_X J)(JX) = G(X,JX) = 0$ where $G(X,Y)$ is defined by $G(X,Y) = (\overline{\nabla}_X J)Y$. Thus $G(X,Y) = 0$, for any $X,Y \in D$. Now using the equation $\overline{\nabla}_Y JX = J\overline{\nabla}_Y X$ with the help of $G(X,Y) = 0$, we get $\nabla_Y JX + h(JX, Y) = J\nabla_Y X + Jh(X,Y)$. Then taking inner product with $Z \in D^{\perp}$, we have $h(X,Y) = 0$ for $X,Y \in D$.

Now we shall prove the following two theorems.

Theorem (1): In S^6 there does not exist a 4-dimensional CR-submanifold with parallel distribution D and constant curvature.

Theorem (2): Let M be a 4-dimensional CR-submanifold in S^6 with parallel holomorphic distribution D. Then the integral submanifold of D is totally geodesic.

proof of theorem (1); Since M is proper $\dim D = \dim D^{\perp} = 2$.

So let $\{e, Je, \xi, \eta\}$ be a frame field for TM and $\{J\xi, J\eta\}$ be a frame field for the normal bundle $\nu = JD^{\perp}$ where $e \in D$ and $\xi, \eta \in D^{\perp}$. We know that $h(X,Y) = 0$ for X,Y in D. Then using Gauss equation (1.3) we get

$$R(e,Je,Je,e) = c,\ R(e,\xi,\xi,e) = \bar{c} - \| h(e,\xi) \|^2$$

$$R(Je,\xi,\xi,Je) = c - \| h(Je,\xi) \|^2,\ R(e,\eta,\eta,e) = c - \| h(e,\eta) \|^2$$

$$R(Je,\eta,\eta,Je) = c - \| h(Je,\eta) \|^2,$$

where c is constant and equal the curvature of S^6. Since M is of constant curvature we have

$$h(e,\xi) = h(Je,\xi) = h(e,\eta) = h(Je,\eta) = 0$$

or equivalently $h(X,Z) = 0$ for $X \varepsilon D$, $Z \varepsilon D^{\perp}$.
Now using $h(X,Z) = 0$ in the equation $\overline{\nabla}_X JZ = J\overline{\nabla}_X Z + G(X,Z)$ and with the help of equations (1.1), (1.2) we get $g(G(X,Z),W) = 0$ for $X \varepsilon D$, Z, $W \varepsilon D^{\perp}$. *i.e.* $g(G(Z,W),X) = 0$. From this we have $G(Z,W)\ \varepsilon D^{\perp}$. But since $g(G(\xi,\eta),\xi) = g(G(\xi,\eta),\eta) = 0$, we have $G(\xi,\eta) \varepsilon D$. Therefore we also have $G(Z,W)\ \varepsilon D$. *i.e.* $G(Z,W) = 0$. From this we get the equation $\overline{\nabla}_W JZ = J\overline{\nabla}_W Z$. Now this last equation with the help of equations (1.1), (1.2) gives $g(\nabla_W Z, JX) = 0$ for any $X \varepsilon D$. *i.e.* $\nabla_W Z \varepsilon D^{\perp}$, $W \varepsilon D^{\perp}$. Also $g(Y,Z) = 0$ and $\nabla_X Y \varepsilon D$ give $\nabla_X Z \varepsilon D^{\perp}$. Therefore $\nabla_V Z \varepsilon D^{\perp}$ for all $V \varepsilon \mathfrak{X}(M)$.

Using $g(X,Z) = 0$ we also get $\nabla_V X \varepsilon D$, for $X \varepsilon D$, $V \varepsilon \mathfrak{X}(M)$. This means that M is a CR-product, a contradiction (Sekigawa 1984).

Proof of theorem (2):

The proof is an immediate consequence of the difinition of parallel distribution. In fact since the holomorphic distribution D is parallel we have for $X, Y \varepsilon D$ $\nabla_X Y \varepsilon D$. This implies that the holomorphic distribution D is integrable.

Then it follows that $h(X,Y) = 0$, for any $X, Y \varepsilon D$. Therefore the integrable submanifold of the holomorphic distribution D is totally geodesic.

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مانفولدات كوشي ـ ريمان الجزئية ذات الأبعاد الأربعة في الكرة ذات الأبعاد الستـة

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لقد جذبت الكرة ذات الابعاد الستة S^6 انتباه كثير من الباحثين الريـاضيين في الهندسة التفاضلية وبالذات أولئك العاملين في مجال هندسة ريمان.

ويرجع السبب في ذلك إلى أن هذه الكرة لها خاصية كهلر التقريبية وليست خاصية كهلر فقط.

هذا الاهتمام أدى بـدورة إلى دراسة المـانفولـدات الجزئيـة في S^6. وتنقسم هذه المانفولدات الجزئية إلى ثلاثة أقسام وهي المانفولدات الجـزئية الحقيقيـة كلياً والمانفولدات الجزئية المركبة ثم مانفولدات كـوشي ريمان الجـزئية. وتمثـل الأخيرة تعمياً للقسمين الأولين.

من المعروف أنه لا توجد هناك مانفولدات جزئية مركبة ذات ابعاد أربعة في S^6 ويرجع اثبات هذه النظرية إلى عالم الرياضيات الفرد قري (Alfred Gray).

من الواضح أيضاً أنه لا توجد هنـاك مانفـولدات جـزئية حقيقيـة كلياً ذات أربعة ابعاد في S^6، وهذا يرجع إلى أن أبعاد الكرة S^6 ستة. ومن ثم بقي لنـا من المانفولدات الجزئية ذات الاربعة أبعاد في S^6 تلك التي تكون من طـابع كـوشي ـ ريمان.

في هـذا البحث درسنا هـذه المانفـولدات الجـزئية الأخيـرة . من المعلوم أن هنـاك توزيعـين يشكلان المجمـوع المباشر لفضـاء المماس بـالنسبة لمـانفولـدات كوشي ـ ريمان الجزئية . وهذان التوزيعان هما التوزيع الحقيقي والتوزيع المركب .

في هـذه الدراسـة افترضنـا أن التـوزيـع المـركب D لـه خـاصيـة التـوازي واستطعنا بناءاً على ذلك أن نثبت نظريتـين تتعلقان بمـانفولـدات كوشي ـ ريمـان الجزئية M ذات الأبعاد الأربعة في S^6 .

النظرية الأولى تنص عـلى عدم وجـود مانفـولدات كـوشي ريمان الجـزئية M ذات انحناء ثابت في S^6 إذا كان التوزيع D متوازياً .

أما النظريـة الثانيـة فتنص على أن المـانفولـد الجزئي التكـاملي للتـوزيع D يكون جيودسك كلي .