

## **Fuzzy Ideals and Filters on Lattices**

**Marouf A. Samhan and Fawzi A. Al-Thukair**

*Department of Mathematics, College of Science,  
King Saud University, P.O.Box 2455,  
Riyadh 11451, Saudi Arabia*

**ABSTRACT.** In this paper we study fuzzy convex sublattice, fuzzy ideals (filters), fuzzy prime ideals (filters) on a lattice and give a characterization of these concepts by their level sets. The fuzzy convex sublattices are represented in terms of fuzzy ideals and filters. Fuzzy maximal ideals (filters) are introduced, and in a special distributive lattices, the fuzzy maximal ideals (filters) are characterized in relation to fuzzy prime ideals (filters).

**Keywords:** Lattice, fuzzy ideal, fuzzy filter, fuzzy prime, fuzzy maximal, fuzzy convex.

### **1. Introduction**

In (Zadeh 1965) the concept of fuzzy sets was introduced, and Rosenfeld in (Rosenfeld 1971) applied the concept to the theory of groups. Since then many papers have been written either to introduce a new concept of a fuzzy algebraic structure or to prove more results on the known ones. In (Yuan and Wangming 1990) the concept of fuzzy ideals in a distributive lattice was introduced. In this paper we study these concepts in more detail, we characterize fuzzy prime ideals (filters) via their level sets, introduce the concept of fuzzy convex sublattice and represent it in terms of fuzzy ideals and filters, fuzzy analogue of the prime ideal theorem is proved.

We also introduce the concept of fuzzy maximal ideals (filters) and prove that in a general lattice fuzzy maximal ideals (filters) are fuzzy prime ideals (filters), we also show that in a generalized Boolean algebra these two concepts are the same. We finally give fuzzy analogue theorems characterizing fuzzy maximal ideals (filters) in Boolean algebras and pseudocomplemented distributive lattices.

Throughout this paper  $L = \langle L, +, \cdot, 0, 1 \rangle$  denotes a lattice with least element 0 and greatest element 1 and fuzzy subsets of  $L$  are maps  $\mu : L \rightarrow [0, 1]$ . For  $a, b \in [0, 1]$ ,  $a \vee b = \max\{a, b\}$  and  $a \wedge b = \min\{a, b\}$ .

## 2. Convex Fuzzy Sublattices

*Definition 2.1.* (Bo and Wangming 1990). A fuzzy subset  $\mu$  of a lattice  $L$  is called a fuzzy sublattice of  $L$  if

$$\mu(x + y) \wedge \mu(xy) \geq \mu(x) \wedge \mu(y) \quad \forall x, y \text{ in } L.$$

*Proposition 2.2.* (Samhan and Al-Thukair 1992). Let  $\mu$  be a fuzzy subset of  $L$ . Then  $\mu$  is a fuzzy sublattice of  $L$  iff the nonempty  $\alpha$ -level sets  $\mu_\alpha = \{x \text{ in } L : \mu(x) \geq \alpha\}$  are sublattices of  $L$  for each  $\alpha \in [0, 1]$ .  $\Delta$

*Definition 2.3.* (Grätzer 1978). A sublattice  $K$  of a lattice  $L$  is called convex sublattice of  $L$  if  $\forall a, b \in K, \forall c \in L$ , if  $a \leq c \leq b$ , then  $c \in K$ .

*Definition 2.4.* A fuzzy sublattice  $\mu$  of  $L$  is called convex iff  $\forall x, y, z \in L$ , if  $x \leq z \leq y$ , then  $\mu(z) \geq \mu(x) \wedge \mu(y)$ .

*Proposition 2.5.* Let  $\mu$  be a fuzzy sublattice of  $L$ . Then  $\mu$  is convex fuzzy sublattice of  $L$  iff the nonempty  $\alpha$ -level sets  $\mu_\alpha$  are convex sublattices of  $L$  for all  $\alpha \in [0, 1]$ .

*Proof.* Assume first that  $\mu$  is a convex fuzzy sublattice of  $L$ . Let  $x, y \in \mu_\alpha$  and  $z \in L$  such that  $x \leq z \leq y$ . Since  $\mu$  is convex, we have  $\mu(z) \geq \mu(x) \wedge \mu(y) \geq \alpha$ , so  $z \in \mu_\alpha$  and  $\mu_\alpha$  is a convex sublattice. Conversely; assume  $\mu_\alpha$  is a convex sublattice of  $L$  for all  $\alpha \in [0, 1]$ .

Let  $x, y, z \in L$  be such that  $x \leq z \leq y$ . If  $\mu(z) < \mu(x) \wedge \mu(y)$ , then there is  $\alpha \in [0, 1]$  such that  $\mu(z) < \alpha < \mu(x) \wedge \mu(y)$ . Thus,  $\mu(x), \mu(y) > \alpha$  and so  $x, y \in \mu_\alpha$ . By the convexity of  $\mu_\alpha$  we have  $z \in \mu_\alpha$  and so  $\mu(z) \geq \alpha$  a contradiction. Hence  $\mu(z) \geq \mu(x) \wedge \mu(y)$  and  $\mu$  is a convex fuzzy sublattice of  $L$ .  $\Delta$

The proofs of the following two lemmas are straightforward.

*Lemma 2.6.* If  $\mu$  and  $\nu$  are fuzzy subsets of  $L$ , then  $(\mu \cap \nu)_\alpha = \mu_\alpha \cap \nu_\alpha \quad \forall \alpha \in [0, 1]$ .

*Lemma 2.7.* If  $\{\mu_i : i \in I\}$  is a family of convex fuzzy sublattices of  $L$ , then so is  $\bigcap_{i \in I} \mu_i$ .

*Definition 2.8.* (Bo and Wangming 1990). A fuzzy subset  $\mu$  of  $L$  is a fuzzy ideal (filter) of  $L$  if  $\forall x, y \in L$ ,

- (i)  $\mu$  is a fuzzy sublattice of  $L$ ;  
(ii) if  $x \leq y$ , then  $\mu(x) \geq \mu(y)$  ( $\mu(x) \leq \mu(y)$ ).

*Proposition 2.9.* Let  $\mu$  be a fuzzy subset of  $L$ . Then  $\mu$  is a fuzzy ideal (filter) of  $L$  iff the nonempty  $\alpha$ -level sets  $\mu_\alpha$  are ideals (filters) of  $L \forall \alpha \in [0,1]$ .

*Proof.* We prove the ideal case, because the filter case can be handled in a similar way. Assume first that  $\mu$  is a fuzzy ideal of  $L$  and  $\alpha \in [0,1]$  such that  $\mu_\alpha \neq \emptyset$ . Let  $x, y \in \mu_\alpha$ .

Since  $\mu(x + y) \geq \mu(x) \wedge \mu(y) \geq \alpha$ , we have  $x + y \in \mu_\alpha$ . Also, if  $x \leq y$  and  $y \in \mu_\alpha$ , then  $\mu(x) \geq \mu(y) \geq \alpha$  and so  $x \in \mu_\alpha$ . Hence  $\mu_\alpha$  is an ideal of  $L$ . Conversely; assume  $\mu_\alpha$  is an ideal of  $L \forall \alpha \in [0,1]$ ,  $\mu_\alpha \neq \emptyset$ . Let  $x, y \in L$  and  $\alpha = \mu(x) \wedge \mu(y)$ . Then  $\mu(x), \mu(y) \geq \alpha$ , and so  $x, y \in \mu_\alpha$ . Thus,  $x + y \in \mu_\alpha$ , and so  $\mu(x + y) \geq \alpha = \mu(x) \wedge \mu(y)$ . Also; if  $x \leq y$  and  $\mu(y) = \alpha$ , then  $y \in \mu_\alpha$  and so  $x \in \mu_\alpha$ . Thus;  $\mu(x) \geq \alpha = \mu(y)$  proving that  $\mu$  is a fuzzy ideal of  $L$ .  $\Delta$

We now characterize convex fuzzy sublattices in terms of fuzzy ideals and fuzzy filters.

*Proposition 2.10.* If  $\mu$  is a fuzzy ideal of  $L$  and  $\nu$  is a fuzzy filter of  $L$  with  $\mu \cap \nu \neq \emptyset$ , then  $\mu \cap \nu$  is a convex fuzzy sublattice of  $L$ .

*Proof.* By Lemma 2.6,  $(\mu \cap \nu)_\alpha = \mu_\alpha \cap \nu_\alpha$ , and by Proposition 2.9,  $\mu_\alpha$  is an ideal and  $\nu_\alpha$  is a filter of  $L$ . By Lemma 6, page 19 of (Grätzer 1978) we have  $\mu_\alpha \cap \nu_\alpha$  is a convex sublattice of  $L$ , and so by Proposition 2.5,  $\mu \cap \nu$  is a convex fuzzy sublattice of  $L$ .  $\Delta$

*Theorem 2.11.* If  $\mu$  is a convex fuzzy sublattice of  $L$ , then there are unique fuzzy ideal  $\gamma$  and fuzzy filter  $\beta$  such that  $\mu = \gamma \cap \beta$ .

*Proof.* By proposition 2.5, we know that the  $\alpha$ -level sets  $\mu_\alpha$  are convex sublattices of  $L$ , for all  $\alpha \in [0,1]$ . By Lemma 6, page 19 of (Grätzer 1978), there are unique ideals  $A_\alpha$

and filters  $B_\alpha$  of  $L$  such that  $\mu_\alpha = A_\alpha \cap B_\alpha$ . Note that  $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$  if  $\alpha_1 \geq \alpha_2$ . By the construction of  $A_\alpha$  and  $B_\alpha$ , it is obvious to see that if  $\alpha_1 \geq \alpha_2$ , then  $A_{\alpha_1} \subseteq A_{\alpha_2}$  and  $B_{\alpha_1} \subseteq B_{\alpha_2}$ .

We construct fuzzy subsets  $\gamma$  and  $\beta$  as follows:

$$\gamma(x) = \sup_{x \in A_\alpha} \alpha$$

$$\beta(x) = \sup_{x \in B_\alpha} \alpha$$

Note that for  $t \in [0,1]$ , the level sets

$$\gamma_t = \bigcap_{\alpha \leq t} A_\alpha$$

$$\beta_t = \bigcap_{\alpha \leq t} B_\alpha.$$

Hence  $\gamma_t, \beta_t$  are ideals, filters respectively. By Proposition 2.9 we conclude that  $\gamma$  is a fuzzy ideal and  $\beta$  is a fuzzy filter.

It remains to show that  $\mu = \gamma \cap \beta$ . Note that

$$\begin{aligned} \mu(x) &= \sup_{x \in \mu_\alpha} \alpha \\ &= \sup_{x \in A_\alpha \cap B_\alpha} \alpha \\ &= \sup_{x \in A_\alpha} \alpha \wedge \sup_{x \in B_\alpha} \alpha \\ &= \gamma(x) \wedge \beta(x) \end{aligned}$$

Thus,  $\mu = \gamma \cap \beta$ .

To prove uniqueness suppose that

$$\mu = \delta \cap \rho,$$

where  $\delta$  is a fuzzy ideal and  $\rho$  is a fuzzy filter.

For each  $\alpha \in [0,1]$ , we have

$$\mu_\alpha = \delta_\alpha \cap \rho_\alpha,$$

where  $\delta_\alpha$  are ideals and  $\rho_\alpha$  are filters. By the uniqueness of this decomposition in Lemma 6 of (Grätzer 1978), we have  $\delta_\alpha = A_\alpha$  and  $\rho_\alpha = B_\alpha$ .

$$\text{Now } \delta(x) = \sup_{x \in \delta_\alpha} \alpha = \sup_{x \in A_\alpha} \alpha = \gamma(x)$$

$$\rho(x) = \sup_{x \in \rho_\alpha} \alpha = \sup_{x \in B_\alpha} \alpha = \beta(x)$$

This shows uniqueness and concludes the proof. Δ

We finish this section by showing that the fuzzy congruence classes of a fuzzy congruence relation of a lattice  $L$  are convex fuzzy sublattices.

By a fuzzy congruence relation on a lattice  $L$  we mean a fuzzy relation  $\theta: L \times L \rightarrow [0,1]$  satisfying the following conditions  $\forall x,y,z, u \in L$

- (i)  $\theta(x,x) = 1$ ;
- (ii)  $\theta(x,y) = \theta(y,x)$ ;
- (iii)  $\theta(x,y) \geq \theta(x,z) \wedge \theta(z,y)$ ;
- (iv)  $\theta(x + y, z + u) \geq \theta(x,z) \wedge \theta(y,u)$ ;
- (v)  $\theta(xy, zu) \geq \theta(x,z) \wedge \theta(y,u)$ .

*Definition 2.12.* (Samhan and Al-Thukair 1992). Let  $L$  be a lattice,  $x \in L$  and  $\theta$  a fuzzy congruence relation on  $L$ . The fuzzy congruence class determined by  $x$  and  $\theta$ , denoted by  $x\theta$  is the fuzzy subset of  $L$  defined by

$$x\theta(y) = \theta(x,y) \quad \forall y \in L.$$

*Theorem 2.13.* With that above notation,  $x\theta$  is a convex fuzzy sublattice of  $L$ .

*Proof.* Let  $y, z \in L$ . Then

$$\begin{aligned} x\theta(y + z) &= \theta(x, y + z) = \theta(x + x, y + z) \\ &\geq \theta(x,y) \wedge \theta(x,z) = x\theta(y) \wedge x\theta(z). \end{aligned}$$

Also;

$$\begin{aligned} x\theta(yz) &= \theta(x,yz) = \theta(xx,yz) \\ &\geq \theta(x,y) \wedge \theta(x,z) = x\theta(y) \wedge x\theta(z). \end{aligned}$$

Finally, if  $y \leq u \leq z$ , then

$$\theta(xu, u) = \theta(xu, uz) \geq \theta(x,z) \wedge \theta(u, u) = \theta(x,z).$$

so;

$$\begin{aligned} \theta(x,u) &= \theta(u x + x, u + y) \\ &\geq \theta(u x, u) \wedge \theta(x,y) \\ &\geq \theta(x,z) \wedge \theta(x,y) \end{aligned}$$

Thus;  $x\theta(u) \geq x\theta(z) \wedge x\theta(y)$  proving that  $x\theta$  is a convex fuzzy sublattice of  $L$ . Δ

### 3. Fuzzy Prime Ideals and Filters

*Definition 3.1.* Let  $\mu$  be a fuzzy ideal (filter) of  $L$ . Then  $\mu$  is called a fuzzy prime ideal (filter) of  $L$  if

- (i)  $\mu$  is nonconstant, and
- (ii)  $\mu(xy) \leq \mu(x) \vee \mu(y)$  ( $\mu(x + y) \leq \mu(x) \vee \mu(y)$ )  $\forall x, y \in L$ .

The following theorem gives the relationship between fuzzy prime ideals (filters) and their level sets.

*Theorem 3.2.* Let  $\mu$  be a nonconstant fuzzy ideal (filter) of  $L$ . Then  $\mu$  is fuzzy prime ideal (filter) of  $L$  iff the nonempty  $\alpha$ -level sets  $\mu_\alpha$  are prime ideals (filters) of  $L$  for each  $\alpha \in [0, 1]$ .

*Proof.* We prove the theorem for ideals (the filter case is handled in a similar way). Assume first that  $\mu$  is a fuzzy prime ideal of  $L$  and  $\alpha \in [0, 1]$  such that  $\mu_\alpha \neq \emptyset$ . Let  $x, y \in L$  and  $xy \in \mu_\alpha$ . Then  $\mu(x) \vee \mu(y) \geq \mu(xy) \geq \alpha$  and so either  $\mu(x) \geq \alpha$  or  $\mu(y) \geq \alpha$ , and hence either  $x \in \mu_\alpha$  or  $y \in \mu_\alpha$ . Therefore,  $\mu_\alpha$  is a prime ideal of  $L$ . Conversely; assume that  $\mu_\alpha$  is a prime ideal of  $L \forall \alpha \in [0, 1]$ , where  $\mu_\alpha \neq \emptyset$ . Let  $x, y \in L$  we will show that  $\mu(xy) \leq \mu(x) \vee \mu(y)$ . Suppose in the contrary that  $\mu(xy) > \mu(x) \vee \mu(y)$ . Then there is  $\alpha \in [0, 1]$  such that  $\mu(xy) > \alpha > \mu(x) \vee \mu(y)$ . Thus,  $xy \in \mu_\alpha$ , and as  $\mu_\alpha$  is prime ideal of  $L$  we must have either  $x \in \mu_\alpha$  or  $y \in \mu_\alpha$ , and this imply that either  $\mu(x) \geq \alpha$  or  $\mu(y) \geq \alpha$ . But then  $\mu(x) \vee \mu(y) \geq \alpha$  a contradiction.  $\Delta$

The following theorem is the fuzzy version of the prime ideal theorem

*Theorem 3.3.* Let  $L$  be a distributive lattice,  $\mu$  is a nonconstant fuzzy ideal and  $\nu$  is a nonconstant fuzzy filter of  $L$  such that  $\mu \cap \nu = \emptyset$ . Then there is a fuzzy prime ideal  $\beta$  of  $L$  such that  $\mu \subseteq \beta$  and  $\beta \cap \nu = \emptyset$ .

*Proof.* The proof of this theorem is along a similar construction used in the proof of *Theorem 2.11*. For  $\alpha \in [0, 1]$ , we have  $\mu_\alpha$  and  $\nu_\alpha$  are ideals and filters of  $L$  respectively, with  $\mu_\alpha \cap \nu_\alpha = \emptyset$ . By Theorem 1 of (Balbes and Dwinger 1974), there are prime ideals  $P_\alpha$  of  $L$  such that  $\mu_\alpha \subseteq P_\alpha$  and  $P_\alpha \cap \nu_\alpha = \emptyset$ . Note that  $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$  if  $\alpha_1 \geq \alpha_2$  and by the proof of *Theorem 1* (Balbes and Dwinger 1974) we have  $P_{\alpha_1} \subseteq P_{\alpha_2}$ .

$$\text{Define } \beta(x) = \begin{cases} \sup_{x \in P_\alpha}, & \text{if such } P_\alpha \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

For  $t \in [0,1]$  note that

$$\beta_t = \bigcap_{\alpha \leq t} P_\alpha,$$

hence  $\beta_t$  is a prime ideal which implies that  $\beta$  is a fuzzy prime ideal by *Theorem 3.2*.

Note that  $\mu(x) = \sup_{x \in \mu_\alpha} \alpha \leq \sup_{x \in \mu_\alpha} \alpha$  (because  $\mu_\alpha \subseteq P_\alpha$ ) thus  $\mu(x) \leq \beta(x)$ .

It remains to show that  $\beta \cap \nu = \emptyset$  which is obvious. Δ

#### 4. Fuzzy Maximal Ideals and Filters

*Definition 4.1.* Let  $\mu$  be a fuzzy ideal (filter) of  $L$ . Then  $\mu$  is called a fuzzy maximal ideal (filter) if  $\mu_1 = \{x \in L : \mu(x) = 1\}$  is a maximal ideal (filter) of  $L$ .

Note that if  $\mu$  is a fuzzy maximal ideal (filter) of  $L$ , then  $\mu(0) = 1$  ( $\mu(1) = 1$ ).

*Proposition 4.2.* If  $\mu$  is a fuzzy maximal ideal (filter) of  $L$  and  $\lambda$ , is any fuzzy ideal (filter) of  $L$  such that  $\mu \subseteq \lambda$ , then either  $\mu_1 = \lambda_1$  or  $\lambda = 1$ .

*Proof.* Since  $\mu_1$  is a maximal ideal of  $L$ , and  $\lambda_1$  is an ideal of  $L$  with  $\mu \subseteq \lambda$ , then we must have  $\mu_1 \subseteq \lambda_1$ , for if  $x \in \mu_1$ , then  $1 = \mu(x) \leq \lambda(x)$ , and so  $x \in \lambda_1$ . Hence by maximality of  $\mu_1$  we must have either  $\mu_1 = \lambda_1$  or  $\lambda_1 = L$ .

The filter case is handled similarly. Δ

The proof of the following theorem is along the lines of *Theorem 3.4* in (Malik and Mordeson 1991) and so we omit the proof.

*Theorem 4.3.* If  $\mu$  is a fuzzy maximal ideal (filter) of  $L$ , then  $|\text{Im}(\mu)| = 2$ .

*Theorem 4.4.* If  $\mu$  is a fuzzy maximal ideal (filter) of  $L$ , then  $\mu$  is a fuzzy prime ideal (filter) of  $L$ .

*Proof.* Assume that  $\mu$  is a fuzzy maximal ideal of  $L$ . Then by *Theorem 4.3*, we have  $\text{Im}(\mu) = \{\alpha, 1\}$ ,  $\alpha < 1$ . Now as  $\mu_1$  is a maximal ideal we have  $\mu_1$  is a prime ideal. Let  $x, y \in L$  we must show that

$$\mu(xy) \leq \mu(x) \vee \mu(y) \quad (1)$$

If  $x, y \in \mu_1$ , then  $\mu(x) = \mu(y) = 1$  hence (1) holds.

If  $x \notin \mu_1$ ,  $y \in \mu_1$ , then  $\mu(x) \vee \mu(y) = 1$  and (1) holds.

If  $x \notin \mu_1$ ,  $y \notin \mu_1$ , then  $xy \notin \mu_1$  and we have  $\mu(x) = \mu(y) = \mu(xy) = \alpha$ , and (1) also holds.

Therefore  $\mu$  is a fuzzy prime ideal of  $L$ . The case of fuzzy maximal filter imply fuzzy prime filter is handled similarly.

In *Theorem 4.6* we show that converse of *Theorem 4.4* is also true in some special lattice  $L$ , however this is not so in general.

*Definition 4.5.* (Balbes and Dwinger 1974). An element  $x \in L$  is relatively complemented if  $x$  is complemented in every interval  $[a, b]$  which contains  $x$ . A generalized Boolean algebra is a distributive lattice with 0 and 1 such that every element is relatively complemented.

*Theorem 4.6.* Let  $L$  be a generalized Boolean algebra and  $\mu$  is a fuzzy prime ideal of  $L$ . Then  $\mu$  is a fuzzy maximal ideal of  $L$ .

*Proof.* Since  $\mu$  is a fuzzy prime ideal, then by *Theorem 3.2*,  $\mu_1$  is prime ideal and by *Corollary 8* of (Balbes and Dwinger 1974) page 69,  $\mu_1$  is a maximal ideal. Therefore  $\mu$  is a fuzzy maximal ideal of  $L$ .

Combining *Theorems 4.4* and *4.6* we get:

*Corollary 4.7.* In a Boolean algebra  $B$ , the fuzzy maximal ideals are precisely the fuzzy prime ideals.

*Theorem 4.8.* Let  $B$  be Boolean algebra and  $\mu$  fuzzy ideal (filter) of  $B$ . Then  $\mu$  is a fuzzy maximal ideal (filter) of  $B$  iff  $\mu(a) = 1$  or  $\mu(a') = 1$  but not both for all  $a \in B$ .

*Proof.* Assume first that  $\mu$  is fuzzy maximal ideal and  $a \in B$ . Then  $\mu_1$  is a maximal ideal of  $B$ . Hence by exercise 11 of (Balbes and Dwinger 1974) page 69, either  $a \in \mu_1$  or  $a' \in \mu_1$  but not both. Hence either  $\mu(a) = 1$  or  $\mu(a') = 1$  but not both.

Conversely; assume that the condition holds. Since  $\mu$  is a fuzzy ideal, we have  $\mu_1$  is an ideal and  $a \in \mu_1$  or  $a' \in \mu_1$  but not both  $\forall a \in B$ . Hence by exercise 11 of (Balbes and

Dwinger 1974) page 69,  $\mu_1$  must be a maximal ideal. Therefore  $\mu$  is a fuzzy maximal ideal. The filter case is handled similarly.  $\Delta$

*Theorem 4.9.* Let  $B$  be a Boolean algebra and  $\mu$  a fuzzy ideal (filter) of  $B$ . Then  $\mu$  is a fuzzy maximal ideal (filter) of  $B$  iff  $\mu(1) < 1$  ( $\mu(0) < 1$ ) and

$$\mu(ab) = \mu(a) \vee \mu(b) (\mu(a+b) = \mu(a) \vee \mu(b)) \text{ for all } a, b \in B.$$

*Proof.* We prove the ideal case and leave the filter case since the proofs are dual. Assume first that  $\mu$  is a fuzzy maximal ideal of  $B$ . Then  $\mu_1$  is a maximal ideal. Hence by *Corollary 3.13* of (Balbes and Dwinger) page 133, we must have  $1 \notin \mu_1$  and  $a \in \mu_1$  iff either  $a \in \mu_1$  or  $b \in \mu_1$ . Thus;  $\mu(1) < 1$  and  $\mu(ab) = \mu(a) \vee \mu(b)$ .

Conversely; Assume that the conditions holds. Then  $1 = \mu(0) = \mu(aa') = \mu(a) \vee \mu(a')$ . Thus;  $\mu(a) = 1$  or  $\mu(a') = 1$ . Now, if  $\mu(a) = \mu(a') = 1$ , then  $\mu(1) = \mu(a+a') \geq \mu(a) \wedge \mu(a') = 1$ , and hence  $\mu(1) = 1$  which is impossible. Thus,  $\mu(a) = 1$  or  $\mu(a') = 1$  but not both, and therefore by *Theorem 4.8*,  $\mu$  is a fuzzy maximal ideal of  $B$ .  $\Delta$

*Definition 4.10.*(Balbes and Dwinger 1974). Let  $L$  be a distributive lattice with 0. An element  $a \in L$  is pseudocomplemented if there is a largest member  $a^*$  of  $L$  with  $aa^* = 0$ .

The element  $a^*$  is called the pseudocomplement of  $a$ . A pseudocomplemented distributive lattice is a distributive lattice with 0 such that every element has a pseudocomplement.

The following theorem holds in the class of pseudocomplemented lattices.

*Theorem 4.11.* Let  $L$  be a pseudocomplemented distributive lattice and  $\mu$  is a non – constant fuzzy subset of  $L$ . Then  $\mu$  is a fuzzy maximal filter of  $L$  if and only if  $\mu$  is a fuzzy prime filter and  $\mu(a + a^*) = 1 \forall a \in L$ .

*Proof.* If  $\mu$  is a fuzzy maximal filter of  $L$ , then by *Theorem 4.4*,  $\mu$  is a fuzzy prime filter. Now;  $\mu_1$  is a maximal filter of  $L$ , thus by *Theorem 2* of (Balbes and Dwinger 1974) page 154,  $a + a^* \in \mu_1$  for all  $a \in L$ , and hence  $\mu(a + a^*) = 1$ . For the converse, assume that  $F$  is a filter of  $L$  such that  $\mu_1 \subset F$ . Suppose that  $a \in F$  and  $a \notin \mu_1$ . Then  $\mu(a) < 1$ . But as  $\mu$  is a fuzzy prime filter, we have  $1 = \mu(a + a^*) \leq \mu(a) \vee \mu(a^*)$ , and so  $\mu(a^*) = 1$ . Hence  $a^* \in \mu_1$ , and so  $a^* \in F$ . Thus,  $0 = aa^* \in F$ . Therefore  $F = L$ .  $\Delta$

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## المثاليات والمرشحات المشوشة على الشبكيات

معروف عبدالرحمن سمحان و فوزي أحمد الذكير

قسم الرياضيات - كلية العلوم - جامعة الملك سعود  
ص.ب: ٢٤٥٥ - الرياض ١١٤٥١ - المملكة العربية السعودية

في عام ١٩٦٥ قدم لطفي زاده (Zadeh 1965) المجموعات المشوشة وقد تلى ذلك عدد كبير من الأبحاث في هذا المجال وباتجاهات مختلفة منها التطبيقي ومنها النظري. إن بحثنا هذا هو من ضمن دراسة البنى الجبرية المشوشة والتي بدأت في عام ١٩٧١ على يد روزنفلد (Rosenfeld 1971) عندما قدم فكرة الزمر الجزئية المشوشة. بعد ذلك قدمت عدة أفكار في مجال البنى الجبرية المشوشة من ضمنها ما قدمه يوان و وانجمنج (Yuan and Wangming 1990) في مجال المثاليات المشوشة في الشبكيات التوزيعية.

ندرس في هذا البحث المثاليات الأولية وكذلك المرشحات المشوشة على الشبكيات من خلال مجموعات المستوى لهذه البنى. كما نقوم باستحداث فكرة الشبكيات الجزئية المحدبة المشوشة وتمثيلها بدلالة المثاليات والمرشحات المشوشة. كذلك نبرهن النظر المشوش لنظرية المثاليات الأولية. بالإضافة إلى ذلك نقدم فكرة المثاليات والمرشحات الأعظمية المشوشة ونبرهن أنه في أي شبكية، كل مثالي أعظمي مشوش هو مثالي أولي مشوش وكذلك الحال بالنسبة

للمرشحات . كما نثبت أنه في الجبر البولياني كل مثالي (مرشح) أعظمي مشوش هو مثالي (مرشح) أولي مشوش . وأخيراً نقدم النظر المشوش لتصنيف المثاليات (المرشحات) الأعظمية في الجبر البولياني والشبكيات التوزيعية شبه المتتمية .