

Global Extrapolation Methods for Fourth Order Parabolic Partial Differential Equations

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ABSTRACT. A family of numerical methods is developed for the numerical solution of fourth order parabolic partial differential equations in one space variable with time dependent boundary conditions.

The methods arise from a two-step, one-parameter family for the numerical solution of systems of second order ordinary differential equations with time dependent source terms. Global extrapolation procedures in time only and in both space and time are discussed.

1. Introduction

It is well known that fourth order parabolic partial differential equations arise in the study of the transverse vibrations of a uniform flexible beam (*e.g.* Gorman 1975). Numerical methods for the approximate solution of such problems have been developed and analysed by a number of authors: chronologically, these include Collatz (1951), Du Fort and Frankel (1953), Crandall (1954), Douglas (1956), Albrecht (1957), Conte (1957), Conte and Royster (1957), Richtmyer (1957), Lees (1961), Evans (1965), Fairweather and Gourlay (1967), Andrade and McKee (1977), Twizell and Khaliq (1983), ter Maten and Sleijpen (1983a,b,c, 1986), ter Maten (1986) and Sleijpen (1986).

In the present paper a family of methods is developed which is second order accurate in space and second or fourth order accurate in time. The family arises from a two-step, one-parameter method for the numerical solution of systems of second order ordinary differential equations with time dependent source terms.

Linear stability analyses, following well known procedures, determine the stability properties of the methods. Global extrapolation is used to improve accuracy.

2. The Associated System of Ordinary Differential Equations

The constant coefficient fourth order parabolic partial differential equation in one space variable, the vibrating beam problem, is given by

$$\frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial^4 u}{\partial x^4} = 0, \mu > 0, X_0 < x < X_1, t > 0 \quad (2.1)$$

where μ is the ratio of flexural rigidity of the beam to its mass per unit length. The time dependent boundary conditions associated with (2.1) will be assumed to have the form

$$u(X_0, t) = p_0(t), \quad u(X_1, t) = p_1(t), \quad t > 0, \quad (2.2)$$

$$\frac{\partial^2 u}{\partial x^2}(X_0, t) = q_0(t), \quad \frac{\partial^2 u}{\partial x^2}(X_1, t) = q_1(t), \quad t > 0 \quad (2.3)$$

and the initial conditions to be given by

$$u(x, 0) = g_0(x), \quad X_0 \leq x \leq X_1, \quad (2.4)$$

$$\frac{\partial u}{\partial t}(x, 0) = g_1(x), \quad X_0 \leq x \leq X_1. \quad (2.5)$$

In (2.2), (2.3), (2.4) and (2.5) the functions $p_0(t)$, $p_1(t)$, $g_0(x)$, $g_1(x)$ are continuous.

The space interval $X_0 \leq x \leq X_1$ will be divided into $N+1$ subintervals each of width h so that $(N+1)h = X_1 - X_0$ and the time variable discretized in steps of length ℓ . The open rectangular region $R = [X_0 < x < X_1] \times [t > 0]$ and its boundary ∂R consisting of the axis $t=0$ and the ordinates $x=X_0$, $x=X_1$ are thus covered by a rectangular mesh of points with coordinates $(X_0 + mh, n\ell)$ where $m=0, 1, \dots, N+1$ and $n=0, 1, 2, \dots$. The theoretical solution of $\{(2.1), (2.2), (2.3), (2.4), (2.5)\}$ at the grid point $(x_m, t_n) \equiv (X_0 + mh, n\ell)$ is, of course, $u(x_m, t_n)$: the theoretical solution of an approximating difference scheme at this grid point will be denoted by U_m^n with $U^n = U(n\ell) = [U_1^n, U_2^n, \dots, U_N^n]^T$, T denoting transpose.

Assuming that $u = u(x, t)$ is sufficiently often differentiable, the space derivative $\partial^4 u / \partial x^4$ in (2.1) may be replaced by the finite difference approximant

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4} &= h^{-4} \{u(x-2h, t) - 4u(x-h, t) + 6u(x, t) - 4u(x+h, t) \\ &+ u(x+2h, t)\} + O(h^2) \end{aligned} \quad (2.6)$$

in which $x = X_0 + mh$ ($m=1,2,\dots,N$) and $t = n\ell$ ($n=0,1,\dots$). For $m=1$ and $m=N$, equation (2.6) introduces the points (X_0-h,t) and (X_1+h,t) which are outside R . It is easy to show, using (2.3), that

$$u(X_0-h,t) = -u(X_0+h,t) + 2p_0(t) + h^2q_0(t) + 0(h^4) \quad (2.7)$$

and

$$u(X_1+h,t) = -u(X_1-h,t) + 2p_1(t) + h^2q_1(t) + 0(h^4) \quad (2.8)$$

and it is these expressions which will be used when (2.6) is used with $x = X_0+h$ and $x = X_1-h$ in (2.1).

Consider the time level $t=n\ell$ and apply (2.1) with (2.6), and (2.7) or (2.8) when necessary, to the N grid points at this value of t . This leads to the system of second order ordinary differential equations, with a time dependent source term, given by

$$\frac{d^2\mathbf{U}}{dt^2} = \mu\mathbf{A}\mathbf{U}(t) - \mu\mathbf{w}(t) \equiv \mathbf{f}(t,\mathbf{U}), \quad t > 0 \quad (2.9)$$

with initial conditions

$$\mathbf{U}(0) = \mathbf{g}_0 \quad \text{and} \quad d\mathbf{U}(0)/dt = \mathbf{g}_1. \quad (2.10)$$

In (2.9) the matrix \mathbf{A} is given by

$$\mathbf{A} = h^{-4} \begin{pmatrix} 5 & -4 & 1 & & & 0 \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & & & & & \\ & & & & & \\ 0 & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix} \quad (2.11)$$

while the vector $\mathbf{w}(t)$ is given by

$$\mathbf{w}(t) = h^{-4}[-2p_0(t) + h^2q_0(t), p_0(t), 0, \dots, 0, p_1(t), -2p_1(t) + h^2q_1(t)]^T. \quad (2.12)$$

3. Numerical Methods

3.1 Introduction and Analyses

Denoting $f(t, U)$ at time $t = n\ell$ by f_n , the family of numerical methods is derived from the two-step, one-parameter family

$$U^{n+1} - 2U^n + U^{n-1} = \ell^2 [af_{n+1} + (1-2a)f_n + af_{n-1}] \quad (3.1)$$

which gives a numerical solution to the linear system (2.9) subject to (2.10).

Theory relating to the numerical solution of ordinary differential equations *e.g.* Lambert 1973) shows that the local truncation error of (3.1) has the form

$$\mathfrak{E}[U(t); \ell] = \left(\frac{1}{12} - a\right)\ell^4 d^4 U/dt^4 + \left(\frac{1}{360} - \frac{1}{12}a\right)\ell^6 d^6 U/dt^6 + \dots \quad (3.2)$$

assuming that $U(t)$ is sufficiently often differentiable with respect to t .

Clearly, (3.1) is a second order numerical method provided $a \neq \frac{1}{12}$: the method is then Numerov's method for the solution of (2.9). It may then be shown that the principal parts of the local truncation errors of numerical methods arising from (3.1) for the solution of {(2.1), (2.2), (2.3), (2.4), (2.5)} have the form

$$\frac{1}{6} \mu h^2 \frac{\partial^6 u}{\partial x^6} + C_i \ell^{i-2} \frac{\partial^i u}{\partial t^i} \quad (3.3)$$

where, from (3.2), $i=4$ when $a \neq \frac{1}{12}$ and $i=6$ when $a = \frac{1}{12}$, and C_i depends on a .

To determine the stability of numerical methods yielded by (3.1) for the solution of fourth order parabolic equations, it is convenient once more to resort to theory associated with methods for second order ordinary differential equations where it is usual to introduce the single linear test problem

$$\ddot{u}(t) = \omega u(t), t > 0; u(0) = g_0, u'(0) = g_1 \quad (3.4)$$

in which \cdot denotes differentiation with respect to t , and $\omega < 0$ is a real constant. Writing $\bar{\ell} = \ell^2 \omega$, so that $\bar{\ell} < 0$, it is easy to show that the interval of absolute stability of the single-equation form of (3.1), when used to solve (3.4), is given by

$$\bar{\ell} \in [-4(1-4a)^{-1}, 0] \quad (3.5)$$

It follows from (3.5), therefore, that $\bar{\ell} \in (-\infty, 0]$ only for $a = \frac{1}{4}$ and that the stability interval is a sub-interval of the negative real number line provided $a < \frac{1}{4}$.

In the context of fourth order parabolic partial differential equations, ω is related to μ and to the eigenvalues $\lambda_s (s=1,2,\dots,N)$ of the matrix A , given by

$$\lambda_s = 16h^{-4} \sin^4[s\pi / \{2(N+1)\}], \quad s=1,2,\dots,N, \quad (3.6)$$

by the relation $\omega = -\mu\lambda$, where λ is some eigenvalue of A . Writing $r = \ell/h^2$ and using (3.5) and (3.6), it may then be shown that the stability interval of the finite difference method arising from (3.1) for a chosen value of a is given by

$$0 \leq r \leq \frac{1}{2} [\mu(1-4a)]^{-1}. \quad (3.7)$$

Confining further discussion of the family of methods to those for which $0 \leq a \leq \frac{1}{4}$, it is evident that there are three methods, based on three particular values of a , which are of interest to the user. These are discussed in the following subsections.

3.2 An Explicit Second Order Method

This method is determined by putting $a=0$ in equation (3.1) and using (2.9) to give the only explicit method of the family, namely

$$\mathbf{U}^{n+1} = (2\mathbf{I} - \mu\ell^2\mathbf{A})\mathbf{U}^n - \mathbf{U}^{n-1} - \mu\ell^2\mathbf{w}^n \quad (3.8)$$

where $\mathbf{w}^n = \mathbf{w}(n\ell)$ is given by (2.13) with $t=n\ell$. The principal part of the local truncation error of this method is given by (3.3) with $q=4$ and $c_4 = \frac{1}{12}$. A conventional stability analysis along the lines described in section 3.1 shows that the method is stable provided $0 \leq r \leq 1/(2\mu^{\frac{1}{2}})$. This method was first developed by Collatz (1951).

3.3 An Unconditionally Stable Second Order Method

Writing $a = \frac{1}{4}$ in (3.1) leads to the method

$$\begin{aligned} (\mathbf{I} + \frac{1}{4}\mu\ell^2\mathbf{A})\mathbf{U}^{n+1} &= (2\mathbf{I} - \frac{1}{2}\mu\ell^2\mathbf{A})\mathbf{U}^n - (\mathbf{I} + \frac{1}{4}\mu\ell^2\mathbf{A})\mathbf{U}^{n-1} \\ &\quad - \frac{1}{4}\mu\ell^2(\mathbf{w}^{n+1} + 2\mathbf{w}^n + \mathbf{w}^{n-1}). \end{aligned} \quad (3.9)$$

The local truncation error of (3.9) is given by (3.3) with $q = 4$ and $C_4 = -\frac{1}{6}$. It is easily verified from (3.7) that the method is unconditionally stable.

The superior stability property of (3.9), compared to (3.8), is partly offset by a slight loss of accuracy and the need to obtain the solution implicitly. Fortunately, the coefficient matrix $\mathbf{I} + \frac{1}{4}\mu\ell^2\mathbf{A}$ is symmetric and quin-diagonal and it is easy to compute the solution vector \mathbf{U}^{n+1} using a standard decomposition algorithm for such a matrix (see Twizell 1984).

3.4 A Fourth Order Method

This method is obtained by putting $a = \frac{1}{12}$ in (3.1). The method is given by

$$\begin{aligned} (I + \frac{1}{12} \mu \ell^2 A) U^{n+1} &= (2I - \frac{5}{6} \mu \ell^2 A) U^n - (I + \frac{1}{12} \mu \ell^2 A) U^{n-1} \\ &- \frac{1}{12} \mu \ell^2 (\mathbf{w}^{n+1} + 10\mathbf{w}^n + \mathbf{w}^{n-1}) \end{aligned} \quad (3.10)$$

and its local truncation error has principal part given by (3.3) with $q=6$ and $C_6 = -\frac{1}{240}$. It follows from (3.7) that the method is stable for r in the interval $0 \leq r \leq \frac{1}{4}(6/\mu)^{\frac{1}{2}}$. The solution using (3.10) is also found by employing a quin-diagonal solver.

3.5 Solution at the First Time Step

It is clear that, using (3.8), (3.9) or (3.10) with $t=\ell$ requires knowledge of $U(\ell)$ which, unlike $U(0)$, is not given explicitly in the initial conditions. It follows from (2.9) and (2.4) that

$$d^2U(0)/dt^2 = -\mu A \mathbf{g}_0 - \mu \mathbf{w}^0, \quad (3.11)$$

where \mathbf{w}^0 is determined from (2.13). Differentiating the differential equation (2.9) and the equation (2.13), and using (2.4) and (2.5) whenever appropriate, gives

$$d^3U(0)/dt^3 = -\mu A \mathbf{g}_1 - \mu \dot{\mathbf{w}}^0, \quad (3.12)$$

$$d^4U(0)/dt^4 = \mu^2 \mathbf{g}_0 + \mu^2 A \mu \mathbf{w}^0 - \mu \ddot{\mathbf{w}}^0, \quad (3.13)$$

$$d^5U(0)/dt^5 = \mu^2 \mathbf{g}_1 + \mu^2 A \mu \dot{\mathbf{w}}^0 - \mu \ddot{\mathbf{w}}^0, \quad (3.14)$$

so that

$$U(\ell) = (I - \frac{1}{2} \mu \ell^2 A) \mathbf{g}_0 + \ell (I - \frac{1}{6} \mu \ell^2 A) \mathbf{g}_1 - \frac{1}{6} \ell^2 \mu (3\mathbf{w}^0 + \ell \dot{\mathbf{w}}^0) + 0(\ell^4) \quad (3.15)$$

or

$$\begin{aligned} U(\ell) &= (I - \frac{1}{2} \mu \ell^2 A + \frac{1}{24} \mu^2 \ell^4 A^2) \mathbf{g}_0 + \ell (I - \frac{1}{6} \mu \ell^2 A + \frac{1}{120} \mu^2 \ell^4 A^2) \mathbf{g}_1 \\ &- \frac{1}{120} \mu \ell^2 \mu (60\mathbf{w}^0 + 20\ell \dot{\mathbf{w}}^0 + 5\ell^2 \ddot{\mathbf{w}}^0 + \ell^3 \ddot{\mathbf{w}}^0) + \frac{1}{120} \mu^2 \ell^4 A (5\mathbf{w}^0 + \ell \dot{\mathbf{w}}^0 + 0(\ell^6)). \end{aligned} \quad (3.16)$$

The approximation given in (3.15) may be used with (3.8) and (3.9) while the approximation given in (3.16) may be used with (3.10).

4. Global Extrapolation Procedures

4.1 Global Extrapolation on Two Time Grids

Consider again the time discretization described in section 2 of the paper, where the time interval $0 \leq t \leq T = Q\ell$ is divided into Q subintervals each of length ℓ . This discretization yields a set of time levels $t_n = n\ell (n=0,1,\dots,Q)$. Suppose now that these time levels are renamed $t_n^{(1)} = n\ell (n=0,1,\dots,Q)$ and that the grid points $(x_m, t_n) \equiv (X_m, t_n^{(1)})$ form a grid named G_1 .

It follows from (3.2) that, after integrating (2.9) to time $t=T=Q\ell$, using (3.1) with an appropriate value of a , the global error, $\epsilon_{Q,1}$, is given by

$$\epsilon_{Q,1} = \ell^q \mathbf{e}_{q,1}(T) + \ell^{q+2} \mathbf{e}_{q+2,1}(T) + \ell^{q+4} \mathbf{e}_{q+4,1}(T) + \dots \quad (4.1)$$

where the functions $\mathbf{e}_{q,1}, \mathbf{e}_{q+2,1}, \dots$ are independent of ℓ (e.g. Verwer and de Vries 1985) and q is the order (in time) of the method.

Suppose now that the interval $0 \leq t \leq T$ is divided into $2Q$ subintervals each of length $\frac{1}{2}\ell$ giving a discretization G_2 consisting of the $2Q+1$ time levels $t_i^{(2)} = \frac{1}{2}i\ell (i=0,1,\dots,2Q)$. Clearly the time points $t_r^{(2)} (r=0,2,4,\dots,2Q)$ of G_2 are coincident with the time points $t_n^{(1)} = n\ell (n=0,1,\dots,Q)$ of G_1 . The notation $\mathbf{V}^r = \mathbf{V}_r(\ell)$ will be used to denote the approximation to $\mathbf{u}(r\ell)$ obtained using (3.1) on grid G_2 : The vector $\mathbf{V}(t)$, like the vector $\mathbf{V}(t)$, has N components because grids G_1 and G_2 each have N points at each time level.

The application of (3.1) to find the solution at time $T=t_Q^{(2)}$ of G_2 generates the global error

$$\epsilon_{2Q,2} = 2^{-q} \ell^q \mathbf{e}_{q,2}(T) + 2^{-q-2} \ell^{q+2} \mathbf{e}_{q+2,2}(T) + 2^{-q-4} \ell^{q+4} \mathbf{e}_{q+4,2}(T) + \dots \quad (4.2)$$

which, like $\epsilon_{Q,1}$, is $O(\ell^q)$, so that $\mathbf{U}(T)$ and $\mathbf{V}(T)$ are both approximations of order q to $\mathbf{u}(T)$.

Consider now the approximation

$$\mathbf{V}^{(E)} = \alpha \mathbf{V} + (1-\alpha) \mathbf{U} \quad (4.3)$$

and the associated global error

$$\mathbf{e}^{(V)} = \alpha \mathbf{e}_{2Q,2} + (1-\alpha) \mathbf{e}_{Q,1} \quad (4.4)$$

It is easy to show that the term in ℓ^q in (4.4) vanishes when the parameter α takes the value

$$\alpha = 2^q / (2^q - 1) \text{ with } 1 - \alpha = -1 / (2^q - 1). \quad (4.5)$$

The global extrapolation carried out using the two grids G_1 and G_2 has thus produced an approximation $\mathbf{V}^{(E)}$ defined by (4.3) which is of order $q+2$ (in time) provided α takes the value given by (4.5). Specifically, the global extrapolation of (3.8) or (3.9) gives the approximation

$$\mathbf{V}^{(E)} = \frac{4}{3} \mathbf{V} - \frac{1}{3} \mathbf{U} \quad (4.6)$$

which leads to a finite difference method which has principal local truncation error $0(h^2 + \ell^4)$ for the solution of $\{(2.1), (2.2), (2.3), (2.4), (2.5)\}$. Note, the same space increment is used with G_1 and G_2 and the increase in accuracy obtained by (4.6) is not related to h . The global extrapolation of (3.10) gives the approximation

$$\mathbf{V}^{(E)} = \frac{16}{15} \mathbf{V} - \frac{1}{15} \mathbf{U} \quad (4.7)$$

which yields a finite difference method with $0(h^2 + \ell^6)$ local truncation error. The stability properties of (3.8), (3.9) and (3.10) are unaffected by global extrapolation.

4.2 Global Extrapolation on Two time- and Two-space Grids

Suppose now that, in addition to halving the time increment, the space increment h is halved also. There are thus $2N+2$ subintervals each of width $\frac{1}{2}h$ in the space interval $X_0 \leq x \leq X_1$ at each of the $2Q+1$ time levels of the time discretization G_2 described in section 4.1. Further renaming the points $(x_m, t_n^{(1)})$, of grid G_1 as $(x_m^{(1)}, t_n^{(1)})$, with $m=0,1,\dots,N+1$ and $n=0,1,\dots,Q$ (thus including boundary points and initial points), the points of G_2 are thus $(x_m^{(1)}, t_r^{(2)})$ with $m=0,1,\dots,N+1$ and $r=0,1,\dots,2Q$. Halving both space and time increments produces a third grid G_3 which has grid points $(x_i^{(2)}, t_r^{(2)})$ with $i=0,1,\dots,2N+2$ and $r=0,1,\dots,2Q$ (assuming that the solution is sought on G_3 at time $T=2Q(\frac{1}{2}\ell)$). The notation $\mathbf{W}^s = \mathbf{W}(s\ell)$ will be used to denote the approximation to $\mathbf{u}(s\ell)$ obtained using (3.1) on grid G_3 . Unlike $\mathbf{U}(t)$ and $\mathbf{V}(t)$ the vector $\mathbf{W}(t)$ has $2N+1$ components because grid G_3 has $2N+1$ points at each time level.

Clearly the space coordinates $x_i^{(2)}$ ($i=0,2,\dots,2N+2$) of G_3 are coincident with the space coordinates $x_m^{(1)}$ ($m=0,1,\dots,N+1$); of G_1 and G_2 and it will be useful to define the operator $I_{\frac{1}{2}h}^h$ by

$$I_{\frac{1}{2}h}^h \mathbf{W}(t) = [\mathbf{W}_2(t), \mathbf{W}_4(t), \dots, \mathbf{W}_{2N}(t)]^T.$$

Effectively, this operator isolates those elements of \mathbf{W} which correspond to the elements of \mathbf{U} and \mathbf{V} .

The application of (3.8), (3.9) or (3.10) to find the solution at time T generates local truncation error vectors (cf (3.3)) which have simplified forms

$$\mathbf{E}_1 = h^2\mathbf{C} + \ell^q\mathbf{D} + 0(h^4 + \ell^{q+2}) \text{ on } G_1, \quad (4.8)$$

$$\mathbf{E}_2 = h^2\mathbf{C} + 2^{-q}\ell^q\mathbf{D} + 0(h^4 + \ell^{q+2}) \text{ on } G_2, \quad (4.9)$$

$$\mathbf{E}_3 = \frac{1}{4}h^2\mathbf{C} + 2^{-q}\ell^q\mathbf{D} + 0(h^4 + \ell^{q+2}) \text{ on } G_3, \quad (4.10)$$

respectively, where \mathbf{C} and \mathbf{D} are independent of h and ℓ , $q=2$ for methods (3.8) and (3.9), and $q=4$ for method (3.10).

Considering the approximation

$$\mathbf{W}^{(E)} = \alpha I_{\frac{1}{2}h}^h \mathbf{W} + \beta \mathbf{V} + (1-\alpha-\beta)\mathbf{U}, \quad (4.11)$$

and the resulting error vector

$$\xi^{(W)} = \alpha I_{\frac{1}{2}h}^h \mathbf{E}_3 + \beta \mathbf{E}_2 + (1-\alpha-\beta)\mathbf{E}_1, \quad (4.12)$$

where α and β are parameters, it may be shown that the terms in h^2 and ℓ^q vanish when

$$(i) \quad \alpha = \frac{4}{3}, \beta=0, 1-\alpha-\beta = -\frac{1}{3} \text{ for (3.8) and (3.9)} \quad (4.13)$$

and

$$(ii) \quad \alpha = \frac{4}{3}, \beta = -\frac{4}{15}, 1-\alpha-\beta = -\frac{1}{15} \text{ for method (3.10)} \quad (4.14)$$

(the reader is referred to an internal Technical Report by Twizell (1987) for a generalization of these findings). The space-time extrapolations of methods (3.8) and (3.9) generate finite difference methods which are $O(h^4 + \ell^4)$ while the space-time extrapolation of (3.10) produces a method which is $O(h^4 + \ell^6)$. It is interesting to note that, because $\beta=0$ in (4.13), the space-time extrapolations of (3.8) and (3.9) use only *two* grids whereas three grids are required in the space-time extrapolation of (3.10).

Modification to the stability condition given by (3.7) is required if the finite difference method associated with a particular value of a is to remain stable on G_3 . This stability criterion must refer to the finest grid (G_3) and so the ratio $r = \ell/h^2$ on the coarse grid G_1 must satisfy the more stringent condition

$$0 \leq r \leq \frac{1}{4}[\mu(1-4a)]^{-\frac{1}{2}} \quad (4.15)$$

if the space-time extrapolation defined by (4.11) is to be stable.

4.3 Comparison of the Implementation Costs of the Extrapolations

The cheapest of the numerical methods to implement is, of course, the explicit method given by (3.8). The additional costs of the other two methods, given by (3.9) and (3.10), are summarised in Table 4.1. These additional costs are due largely to the need to solve a quindagonal linear system (of order N on grids G_1 and G_2 and order $2N+1$ on grid G_3) each time either method is used: like method (3.8), method (3.9) is used $Q-1$ times on G_1 and $2Q-1$ times on G_3 while method (3.10) is used $Q-1$ times on G_1 and $2Q-1$ times on each of G_2 and G_3 . The additional costs of method (3.10) include N extra multiplications and N extra subtractions because $\beta \neq 0$ in (4.11) for this method.

Table 4.1. Additional costs in implementing methods (3.9) and (3.10) in comparison to method (3.8)

method	grids	multiplications or divisions	additions or subtractions
(3.9),(3.10)	G_1 only	$11NQ - 11N - 16Q + 16$	$8NQ - 8N - 13Q + 13$
(3.9),(3.10)	G_1, G_2	$33NQ - 22N - 48Q + 32$	$24NQ - 16N - 39Q + 26$
(3.9)	G_1, G_3	$55NQ - 33N - 26Q + 21$	$40NQ - 24N - 23Q + 18$
(3.10)	G_1, G_2, G_3	$77NQ - 43N - 58Q + 37$	$56NQ - 31N - 49Q + 31$

5. Numerical Results

To observe the behaviour of the three numerical methods (3.8), (3.9) and (3.10), they and their time-only and space-time global extrapolations were tested on the following problem.

Problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\pi^4} \frac{\partial^4 u}{\partial x^4} = 0; \quad 0 < x < 1, \quad t > 0,$$

with initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \cos \pi x; \quad 0 \leq x \leq 1$$

and time-dependent boundary conditions

$$u(0,t) = -u(1,t) = \sin t, \quad \frac{\partial^2 u}{\partial x^2}(0,t) = -\frac{\partial^2 u}{\partial x^2}(1,t) = \pi^2 \sin t; \quad t > 0,$$

which has theoretical solution

$$u(x,t) = \cos \pi x \sin t.$$

The increments h and ℓ were, first of all, given the values 0.1 and $3\pi/400$ respectively, and the solution computed for $t=3\pi/2$ so that 200 applications were required of each two-step method tested. These values of h and ℓ give $r=\ell/h^2 \approx 2.36$ which is well inside the stability interval $0 < r \leq \frac{1}{2} \pi^2 \approx 4.93$ of the second order explicit method (3.8) for this problem, and of the stability interval $0 < r \leq \frac{1}{4} \pi^2 \sqrt{6} \approx 6.04$ for the fourth order implicit method (3.10). The errors $u - \tilde{U}$ in the computed solution \tilde{U} at time $t=3\pi/2$ are contained in Tables 5.1, 5.2 and 5.3 for the numerical methods (3.8), (3.9) and (3.10) respectively. The errors are given for $x=0.1(0.1)0.5$ only, the errors for $x=0.6(0.1)0.9$ being the same in magnitude but opposite in sign to those for $x=0.4(-0.1)0.1$.

Table 5.1. Error $u - \tilde{U}$ at points $x=0.1(0.1)0.5$ when $t=3\pi/2$ using the explicit method (3.8) with $h=0.1$ and $\ell=3\pi/400$

x	one-grid	time-only extrapolation	space-time extrapolation
0.1	0.141(-2)	0.141(-2)	0.425(-4)
0.2	0.197(-2)	0.198(-2)	0.718(-4)
0.3	0.170(-2)	0.179(-2)	0.739(-4)
0.4	0.104(-2)	0.104(-2)	0.469(-4)
0.5	0.305(-17)	-0.111(-16)	0.164(-16)

Table 5.2. Error $u-\tilde{U}$ at points $x=0.1(0.1)0.5$ when $t=3\pi/2$ using the second order implicit method (3.9) with $h=0.1$ and $\ell=3\pi/400$

x	one-grid	time-only extrapolation	space-time extrapolation
0.1	0.142(-2)	0.141(-2)	0.404(-4)
0.2	0.198(-2)	0.198(-2)	0.730(-4)
0.3	0.173(-2)	0.179(-2)	0.787(-4)
0.4	0.104(-2)	0.103(-2)	0.500(-4)
0.5	0.160(-16)	0.192(-15)	0.130(-15)

Table 5.3. Error $u-\tilde{U}$ at points $x=0.1(0.1)0.5$ when $t=3\pi/2$ using the fourth order implicit method (3.10) with $h=0.1$ and $\ell=3\pi/400$

x	one-grid	time-only extrapolation	space-time extrapolation
0.1	0.141(-2)	0.141(-2)	0.428(-4)
0.2	0.197(-2)	0.198(-2)	0.715(-4)
0.3	0.169(-2)	0.179(-2)	0.751(-4)
0.4	0.104(-2)	0.104(-2)	0.489(-4)
0.5	-0.326(-16)	0.543(-16)	0.148(-15)

It is noted that, as indicated in the discussion of local truncation errors, the second order explicit method (3.8) gives slightly better results than the implicit method (3.9). The superiority of the latter method is in its stability properties as it may clearly be used when the former method does not satisfy (3.7) with $a=0$. As the value of ℓ used in the numerical experiment was small, the improvement in accuracy predicted by the local truncation error of (3.10) is not reflected in Table 5.3 where results are quoted only to three significant figures.

Next, the value of ℓ was decreased to $3\pi/800$ and results obtained on grid G_2 of section 4.1. These results together with those on grid G_1 were then combined as indicated by equations (4.6) and (4.7) to give globally extrapolated results (in time) at time $t=3\pi/2$. The errors associated with this extrapolation technique are also reported in Tables 5.1, 5.2 and 5.3 for the three methods (3.8), (3.9) and (3.10). Again, the improvements in accuracy predicted by the theory are not evident because the time steps used on G_1 and G_2 were small.

Finally, keeping the smaller time step $3\pi/800$ and halving the step size in x to 0.05, results were obtained on grid G_3 of section 4.2. In the cases of the two $0(h^2+\ell^2)$ methods given by (3.8) and (3.9), the results obtained on grid G_3 were combined with those obtained on grid G_1 as indicated by (4.11) and (4.13) to give globally extrapolated results which were $0(h^4+\ell^4)$. In the case of the $0(h^2+\ell^4)$ method (3.10) the results obtained on grids G_1 , G_2 and G_3 were combined in the way indicated by (4.11) and (4.14) to give $0(h^4+\ell^6)$ results. The errors relating to the space-time extrapolations are also contained in Tables 5.1, 5.2 and 5.3 for the three methods (3.8), (3.9) and (3.10), respectively. Following the space-time extrapolations there is a marked increase in accuracy though, again, the extra accuracy predicted by the theory for method (3.10) is not evident; this, again, is because the time step used on G_3 is small. It is noted that the ratio $\text{time-step}/(\text{space-step})^2$ on grid G_3 is $3\pi/2 \approx 4.71$ which is just inside the limit indicated by (3.7) with $a=0$ for the explicit method (3.8).

6. Conclusions

A family of numerical methods has been developed in this paper for the numerical solution of fourth order parabolic partial differential equations in one space variable which govern the transverse vibrations of a uniform flexible beam. The boundary conditions were taken to be time dependent.

In developing the numerical methods, the *method of lines* approach was used. The resulting system of second order ordinary differential equations was solved using a one-parameter, linear two-step method giving a family of finite difference methods for the solution of the partial differential equation. One member of the family was seen to be $0(h^2+\ell^4)$, the others to be $0(h^2+\ell^2)$.

A global extrapolation method utilizing two-time grids was developed to increase the accuracies to $0(h^2+\ell^6)$ and $0(h^2+\ell^4)$, respectively. Global extrapolation in both space and time was also considered, increasing the accuracies to $0(h^4+\ell^6)$ and $0(h^4+\ell^4)$, respectively.

The methods were tested on a problem with periodic boundary conditions. It was seen that the extrapolation procedure involving both space and time produced noticeable reductions in error.

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طرق الاستنتاج الشاملة للرتبة الرابعة من القطع المكافئ للمعادلات التفاضلية الجزئية

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استنبطت مجموعة من الطرق العددية من أجل الحلول العددية للمعادلات التفاضلية الجزئية من الرتبة الرابعة ذات القطع المكافئ خلال حيز واحد متغير مع حدود معتمدة على الزمن.

وقد استنبطت هذه الطرق من خطوتين، مجموعة ذات معامل مشتق أوحد للحلول العددية لأنظمة ذات معادلات إعتيادية من الرتبة الثانية مع حدود معتمدة على الزمن. كما نوقشت كل من العمليات الكلية للمد من الخارج من خلال الزمن ومن خلال الزمن والحيز.