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Bayesian Estimation of P(X<Y) for a Bivariate Exponential Distribution

M.S. Abu-Salih[1] and A.A. Shamseldin[2]

[1]Yarmouk University, Irbid, Jordan and
[2]King Saud University, Riyadh, Saudi Arabia

ABSTRACT. We investigate Bayesian estimation of P(X<Y) when X and Y follow the bivariate exponential distribution of Marshall and Olkin with two parameters. Bayes' estimators are derived in case of natural conjugate prior. The Bayes' variances of the estimators are derived. A numerical example is given.

1. Introduction

The problem of estimating $P = P(X < Y)$ has been discussed by several authors when X and Y have certain specified distributions. Downton (1973) derived the minimum variance unbiased estimator (MVUE) of P when X and Y are independent normal variables. Tong (1974) obtained the MVUE of P in the exponential case. Johnson (1975) gave a correction to Tong's result. Beg (1980) derived the MVUE of P for truncation parameter distribution. Abu-Salih and Al-Fayoumi (1986) gave four estimators for P in case of power-function model. This problem originated in the context of reliability of a component of strength Y subjected to a stress X. The component fails if at any time its strength is exceeded by the stress applied to it. This type of reliability model is known as stress-strength model.

Awad *et al.* (1981) have derived three estimators of P when X and Y have a bivariate exponential (BVE) distribution. The BVE distribution was derived by Marshall and Olkin (1976) as a failure model of a two-component system when there exists a positive probability of simultaneous failure of the two components beside the individual failure of each one. In this model, we assume the existance of three independent sources of shocks in the environment. A shock from source (1) destroys component (1) and occurs at an exponential radnom time of failure rate parameter λ_1. A shock from source (2) destroys component (2), and occurs at an exponential random time of parameter λ_2. Finally, a shock from source (3) destroys both components and occurs at an exponential random time of parameter λ_0. We restrict ourselves to the case of two parameters, namely when $\lambda_2 = \lambda_1$. We denote the distribution by BVE $|\lambda_1,\lambda_0$. The joint survival probability function of BVE $|\lambda_1,\lambda_0$ distributin is given by

(1.1)
$$\bar{F}(x,y) = P\ (X > x,\ Y > y)$$

$$= \begin{cases} \exp[-\lambda_1(x+y) - \lambda_0 \max\ (x,y)], & x\geq o,\ y\geq o \\ O & ,\ \text{otherwise} \end{cases}$$

where (λ_1, λ_0) is a realization of the random vector (Λ_1,Λ_0) that assumes the values in the set Ω given by

$$\Omega = \{(\lambda_1,\lambda_0) : o<\lambda_1+\lambda_0<\infty, \quad \lambda_0\geq o, \quad \lambda_1\geq o\}$$

Hence, $P(X<Y) = \iint\limits_{x<y} dF(x,y) = \dfrac{\lambda_1}{\lambda} = P_1$

$$P(X>Y) = \iint\limits_{y<x} dF(x,y) = \frac{\lambda_1}{\lambda} = P_1$$

and $P(X=Y) = 1-P(X<Y)-P(X>Y)$

$$= \frac{\lambda_0}{\lambda} = P_0$$

where $\lambda = 2\lambda_1 + \lambda_0$.

Our objective is to estimate the above probabilities using Bayesian approach.

Ldt $\underset{\sim}{P} = \left(\dfrac{\lambda_1}{\lambda}, \dfrac{\lambda_0}{\lambda}\right)$ be a realization of $\underset{\sim}{P} = \left(\dfrac{\Lambda_1}{\Lambda}, \dfrac{\Lambda_0}{\Lambda}\right)$

In this paper we derive Bayes' estimators of $\left(\dfrac{\Lambda_1}{\Lambda}, \dfrac{\Lambda_0}{\Lambda}\right)$

under the assumption that Λ_1 and Λ_0 are independent apriori and each has a gamma prior density.

2. Likelihood Function

In the sequel was use (T_1, T_2) instead of (X,Y). Let (T_1, T_2) have the bivariate exponential distribution given by (1.1). Denote this distribution by $BVE|\lambda_1,\lambda_0$. Let $\{\underline{T}_j = (T_{1j}, T_{2j})\}_{j=1}^{n}$ denote a random sample of size n from $BVE|\lambda_1,\lambda_0$, and $\{\underline{t}_j = (t_{1j}, t_{2j})\}_{j=1}^{n}$ be the corresponding set of sample values. Let n_0 be the number of observations in the region $\{t_1 = t_2\}$. Denote the random counterpart of n_0 by N_0. It has been shown by Bhattacharyya and Johnson (1973), that the set $(N_0, \Sigma \max (T_{1j}, T_{2j}), \Sigma(T_{1j} + T_{2j}))$ constitutes a set of minimal sufficient statistics for the $BVE|\lambda_1,\lambda_0$ family. Therefore, the likelihood function of a sample of size n from this family is given by:

(2.1)

$$L(\lambda_1,\lambda_0) = [\lambda_1(\lambda_1+\lambda_0)]^{n-n_0} \lambda_0^{n_0} \exp \{-\lambda_1\tau_1-\lambda_0\tau_0\}, \quad (\lambda_1,\lambda_0) \in \Omega$$

$$= 0, \qquad \text{otherwise},$$

where $\tau_1 = \sum_{j=1}^{n} (t_{1j} + t_{2j})$,

$$\tau_0 = \sum_{j=1}^{n} \max (t_{1j}, t_{2j}).$$

(Bemis *et al.* 1972 and Bhattacharyya and Johnson 1973).

We write (2.1) in the form

(2.2)

$$L(\lambda_1,\lambda_0) = \sum_{j=0}^{n-n_0} \binom{n-n_0}{j} . \lambda_1^{n-n_0+j} \lambda_0^{n-j}. \exp\{-\lambda_1\tau_1-\lambda_0\tau_0\}, \quad (\lambda_1,\lambda_0)\in\Omega$$

$$= 0 \qquad , \text{otherwise}$$

3. Bayes' Estimator for $\underline{P}$

In this paper we are interested in the case where we have prior information about λ_1 and λ_0 which can be quantified mathematically by

(3.1)

$$h(\lambda_1,\lambda_0) = g_1(\lambda_1)g_0(\lambda_0),$$

where $g_i(\lambda_i) \propto \lambda_i^{v_i-1} \bar{e}^{\alpha_i\lambda_i}$, i=0,1

and $\quad 0 < \lambda_i < \infty, \; v_i > 0, \; \lambda_i > 0, \; i=0,1.$

It is to be mentioned that the hyperparameters ν_i, α_i, i=0,1 have to be assessed apriori. This assessment assumes that Λ_1 and Λ_0 are independent apriori, and each has a gamma prior density. We note that (3.1) is a conjugate prior for (2.2). Besides this fact, gamma prior densities are "flexible enough to capture almost any kind of prior experience" (Bhattacharyya 1967). For more details on the choice of prior densities on the parameter space which are capable of summerizing the experimenter's prior knowledge, one is referred to Raiffa and Schlaifer (1961). Let $\underline{d} = (\tilde{\underline{d}}, \underline{h})$ where $\underline{d} = (n_0, \tau_0, \tau_1)$ is a realization of the vector

$$(N_0, \sum_{j=1}^{n} \max(T_{1j}, T_{2j}), \sum_{j=1}^{n} (T_{1j}+T_{2j})) \text{ and } \underline{h} = (\nu_i, \alpha_i, i=0,1)$$

are the values of the hyperparameters. From (2.2), (3.1), and using Bayes' theorem we get the joint posterior density function of Λ_1, Λ_0 to be:

(3.2)

$$\pi(\lambda_1, \lambda_0|\underline{d}) = C(\underline{d}). \sum_{j=0}^{n_1} \binom{n_1}{j} \lambda_1^{n_1+\nu_1+j-1}$$

$$\lambda_0^{n+\nu_0-j-1} \exp\{-\lambda_1(\tau_1+\alpha_1) - \lambda_0(\tau_0+\alpha_0)\},$$

$$0 < \lambda_1 < \infty, \quad 0 < \lambda_0 < \infty,$$

$$= 0 \qquad , \text{otherwise}$$

where $n_1 = n - n_0$ and $C(\underline{d})$ satisfies $\int\int \pi(\lambda_1, \lambda_0|\underline{d}) d\lambda_1 d\lambda_0 = 1$.

Inegrating (3.2) with respect to λ_1 and λ_0 and equating the result to one, we get,

(3.3)

$$C^{-1}(\underline{d}) = \sum_{j=0}^{n_1} \binom{n_1}{j} \frac{\Gamma(n_1+\nu_1+j)}{(\tau_1+\alpha_1)^{n_1+\nu_1+j}} \frac{\Gamma(n+\nu_0-j)}{(\tau_0+\alpha_0)^{n+\nu_0-j}}$$

The Bayes' estimator $\hat{P}_0$ of $P_0 = \dfrac{\lambda_0}{\lambda}$, against squared-error loss $L(P_0, \hat{P}_0) = (P_0 - \hat{P}_0)^2$, is given by $\hat{P}_0 = E\left(\dfrac{\Lambda_0}{\Lambda}|\underline{d}\right)$ To evaluate the posterior mean, we first derive the posterior density of

$$V = \frac{\Lambda_0}{\Lambda} = \frac{\Lambda_0}{\Lambda_0 + 2\Lambda_1}$$

Using (3.2) we find the posterior density of V to be:

(3.4)

$$\pi(v|\underline{d}) = C(\underline{d}) = \sum_{j=0}^{n_1} \binom{n_1}{j} . \frac{\Gamma(n+n_1+v_1+v_0)}{2^{n_1+v_1+j}}$$

$$\frac{v^{n+v_0-j-1}\,(1-v)^{n_1+v_1+j-1}}{\left[(\tau_0+\alpha_0)v+\frac{\tau_1+\alpha_1}{2}(1-v)\right]^{(n+n_1+v_1+v_0)}}$$

$$= 0 \qquad , \text{otherwise}$$

Formula (3.4) can be written in the form

(3.5)

$$\pi(v|\underline{d}) = C(\underline{d}) = \sum_{j=0}^{n_1} \binom{n_1}{j}\, 2^{n+v_0-j}\, \frac{\Gamma(n+n_1+v_1+v_0)}{(\tau_1+\alpha_1)^{n+n_1+v_1+v_0}}$$

$$v^{n+v_0-j-1}\,(1-v)^{n_1+v_1+j-1}\,(1-\beta v)^{-(n+n_1+v_1+v_0)}, \quad 0<v<1$$

$$= 0 \qquad , \text{otherwise},$$

where $\beta = \dfrac{(\tau_1+\alpha_1)-2(\tau_0+\alpha_0)}{\tau_1+\alpha_1}$

The density in (3.5) can be put in the closed form:

(3.6)

$$\pi(v|\underline{d}) = C(\underline{d})\, \frac{2^{n_0+v_0}\,\Gamma(n+n_1+v_1+v_0)}{(\tau_1+\alpha_1)^{n+n_1+v_1+v_0}}$$

$$v^{n_0+v_0-1}\,(1-v)^{n_1+v_1-1}\,(1+v)^{n_1}.$$

$$(1-\beta v)^{-(n+n_1+v_1+v_0)}, \quad 0<v<1$$

$$= 0 \qquad , \text{otherwise}.$$

Using (3.5) and the integration formula:

$$\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\beta x)^{-a}dx =$$

$$B(b,c-b)\,{}_2F_1(a,b;c;\beta),\ \mathrm{Re}\,c>\mathrm{Re}\,b>0,\ |\beta|<1,$$

where $B(b,c-b) = \dfrac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}$, and

${}_2F_1(a,b;c;z)$ is the hypergeometric function. Erdelyi *et al.* (1953), we easily get the Bayes' estimator of P_0 to be:

(3.7)

$$\hat{P}_0 = E(V|\underline{d}) = C(\underline{d}) \sum_{j=0}^{n_1} \binom{n_1}{j} 2^{n+\nu_0-j}.$$

$$\frac{\Gamma(n+n_1+\nu_1+\nu_0)}{(\tau_1+\alpha_1)^{n+n_1+\nu_1+\nu_0}} B(n+\nu_0-j+1, n_1+\nu_1+j).$$

$${}_2F_1(n+n_1+\nu_1+\nu_0,\ n+\nu_0-j+1;\ n+n_1+\nu_1+\nu_0+1;\beta).$$

The variance of $\hat{P}_0$ is equal to the marginal posterior variance of $\dfrac{\Lambda_0}{\Lambda}$ and is given by:

(3.8)

$$\mathrm{Var}(\hat{P}_0) = \mathrm{Var}\left(\frac{\Lambda_0}{\Lambda}|\underline{d}\right) = E(V^2|\underline{d}) - \{\hat{P}_0\}^2$$

$$= C(\underline{d}) \sum_{j=0}^{n_1} \binom{n_1}{j} 2^{n+\nu_0-j}. \frac{\Gamma(n+n_1+\nu_1+\nu_0)}{(\tau_1+\alpha_1)^{n+n_1+\nu_1+\nu_0}}$$

$$B(n+\nu_0-j+2,\ n_1+\nu_1+j).$$

$${}_2F_1(n+n_1+\nu_1+\nu_0,\ n+\nu_0-j+2;\ n+n_1+\nu_1+\nu_0+2;\beta) - \{\hat{P}_0\}^2$$

Since $P_1 = \frac{1}{2}(1-P_0)$, the Bayes' estimator of P_1 against squared error loss is

(3.9)

$$\hat{P}_1 = \tfrac{1}{2}(1 - \hat{P}_0)$$

The variance of $\hat{P}_1$ and covariance of $\hat{P}_0$ and $\hat{P}_1$ are given by:

(3.10)

$$\mathrm{Var}\ (\hat{P}_1) = \tfrac{1}{4}\ \mathrm{Var}\ (\hat{P}_0), \quad \text{and}$$

(3.11)

$$\mathrm{Cov}\ (\hat{P}_0,\hat{P}_1) = -\tfrac{1}{2}\ \mathrm{Var}\ (\hat{P}_0)$$

4. A Numerical Example

A random sample of size 10 of two component parallel systems was simulated. It was assumed that the failure times (T_1, T_2) in each system follows the BVE$|\lambda_1,\lambda_0$. The results are listed in Table 1.

Table 1. Observed values of failure times (T_1,T_2) for 10 simulated BVE$|\lambda_1,\lambda_0$ parallel systems: $\lambda_1 = 2$, $\lambda_0 = 1$

Sample	T_1	T_2
1	0.3252	2.0345
2	0.2077	0.3181
3	0.1200	0.1200
4	0.4144	0.1459
5	0.1693	0.5512
6	0.0838	0.0838
7	0.6834	0.3303
8	0.3592	0.3592
9	0.2525	0.4320
10	0.4319	0.4319

From Table 1, the jointly minimal sufficient statistic $\underline{D} = (N_0, \sum_{j=1}^{10} \max(T_{1j},T_{2j}), \sum_{j=1}^{10} (T_{1j}+T_{2j})$ takes the value $\underline{d}$ = (4, 5.9393, 8.6138). We shall compute the Bayes' estimates for $P_0 = \dfrac{\lambda_0}{\lambda}$ and $P_1 = \dfrac{\lambda_1}{\lambda}$ for the case of a vague prior knowledge ($\alpha_i = 0$, $\nu_i = 0$, $i = 0,1$).

The Bayes' estimators for P_0 and P_1 with respect to the quadratic loss function are given by (3.7) and by (3.9), respectively.

Thus,

$\hat{P}_0 = 0.2232$

$\hat{P}_1 = 0.3844$

The marginal posterior variances and covariances of $\frac{\Lambda_0}{\Lambda}$ and $\frac{\Lambda_1}{\Lambda}$ are computed from formulae (3.8), (3.10) and (3.11), respectively. The results are

$$\text{Var}\left(\frac{\Lambda_0}{\Lambda} \,|\underline{d}\right) = 0.01478$$

$$\text{Var}\left(\frac{\Lambda_1}{\Lambda} \,|\underline{d}\right) = 0.0037$$

$$\text{Cov}\left(\frac{\Lambda_0}{\Lambda}, \frac{\Lambda_1}{\Lambda} \,|\underline{d}\right) = 0.0074$$

5. Conclusions

We have given Bayes' estimators in the case of squared error loss. The results are given in closed form and are simple to calculate. This work can be extended to derive Bayes' estimators with respect to a more complicated loss functions than the squared-error loss.

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تقدير بييز لاحتمال س أصغر من ص في حالة التوزيع الأسي الثنائي

[1]محمد صبحي أبو صالح و [2]عادل شمس الدين

١ - جامعة اليرموك - أربد - الأردن
٢ - جامعة الملك سعود - الرياض - المملكة العربية السعودية

قمنا في هذا البحث باشتقاق تقدير بييز لاحتمال س أصغر من ص عندما يكون المتغيران س، ص خاضعين لتوزيع أسي ثنائي ذي معلمتين وذلك على فرض أنهما مستقلتان وأن التوزيع القبلي لكل منهما هو توزيع جاما.

لقد تم إيجاد التباينات للتقديرات التي استنتجناها وأعطينا مثالاً عددياً يبين سهولة الحسابات.