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Thin Circular Plate Under Normal Paraboloidal Loading over a Concentric Ellipse and Symmetrically Supported at Four Points

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ABSTRACT. Exact expressions in infinite series are given for the small deflections, moments and shears in a thin isotropic circular plate subject to normal paraboloidal loading distributed over the area of a concentric ellipse and supported at the four corners of a concentric rectangle whose sides are parallel to the axes of the ellipse. Limiting cases are investigated. Numerical results are presented in the form of tables and graphs illustrating the variation of the deflection, moments and shears along various radii in the first quadrant of the plate.

1. Introduction

Thin circular plates supported along their edges or along concentric circles or at a discrete number of points and subject to various distributions of normal pressures either over the entire plate or part of it have been extensively studied by many investigators. Symmetrically loaded thin circular plates supported at equally spaced points on a concentric circle have been studied by Kirstein *et al.* (1966), Kirstein and Woolley (1967, 1968) and their experimental results compare favorably with Bassali's theory (1957). Circular plates on multipoint supports were also analysed by Yu and Pan (1966), Vaughan (1970), and Williams and Birnson (1974). The deflection surface of a thin circular annulus supported at equispaced points along a concentric circle and subject to symmetrical loading distributed either over its entire surface or over the area of a concentric circular annulus was obtained by one of the authors (Bassali 1984, 1986a). Two recent papers (Bassali 1986 b,c) deal with the two cases of uniform or uniformly varying normal loadings over the area of an ellipse concentric with the circular plate which is supported at the four vertices of a concentric rectangle whose sides are parallel to the axes of the ellipse, the boundary of the plate being free. Taking the conditions for a free boundary in the complex form used by Bassali (1957, 1958), Adeboye and Nassif (1979) found the

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complex potentials for a concentrically supported circular plate parabolically loaded over a concentric ellipse. It is worthy of mentioning here that the method of complex potentials was applied by Frischbier and Lucht (1970) to deal with the case of a clamped circular plate subject to uniform normal loading over a concentric regular polygon.

The problem considered in the present paper is a continuation of the two previous problems (Bassali 1986 b, c) but with paraboloidal loading over the area of the concentric ellipse. The limiting cases in which the radius of the plate $\rightarrow \infty$ or the eccentricity of the ellipse $\rightarrow 0$ or its minor axis $\rightarrow 0$ are discussed. Numerical results and curves are presented for two representative special problems.

2. Mathematical Formulation of the Problem

Let C denote the boundary of a thin circular plate of constant thickness, centre 0, radius c and flexural rigidity D. We assume that $z = x + iy = re^{i\theta}$ is the complex variable of any point N in the mid-plane of the plate and that Γ denotes the boundary of the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \quad (0 \leqslant b \leqslant a \leqslant c). \tag{1}$$

Let the indices 1 and 2 refer to the region inside Γ and that between Γ and C, respectively. It is assumed that:

(a) The intensities of the normal loading on the plate are given by

$$p_1 = p_0 r^2, \quad p_2 = 0 \quad (p_0 \text{ constant}), \tag{2}$$

(b) The plate is supported at the four points P_n ($z_n = se^{i\gamma_n}$, n = 1,2,3,4), where $0 \leqslant s \leqslant c$, $u = s/c$, $\gamma_1 = \gamma$, $\gamma_2 = \pi - \gamma$, $\gamma_3 = \gamma - \pi$, $\gamma_4 = -\gamma$, $0 \leqslant \gamma \leqslant \pi/2$. See Fig. 1.

(c) The boundary C of the plate is free.

If w is the small deflection, measured positively downwards, at the point N it is required to determine w_1 and w_2 which satisfy the following conditions:

(i) $\nabla^4 w_1 = p_0 r^2/D, \quad \nabla^4 w_2 = 0,$ (3a)

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\partial^2}{\partial z \partial \bar{z}} = d^2 + r^{-1}d + r^{-2}d'^2, \quad d = \frac{\partial}{\partial r},$$

$$d' = \frac{\partial}{\partial\theta}, \tag{3b}$$

(ii) the continuity requirements

$$[w]_1^2 = \left[\frac{\partial w}{\partial z}\right]_1^2 = \left[\frac{\partial^2 w}{\partial z \partial \bar{z}}\right]_1^2 = \left[\frac{\partial^3 w}{\partial z^2 \partial \bar{z}}\right]_1^2 = 0 \tag{4}$$

at any point on Γ,

(iii) the two conditions for C to be free (Bassali 1986b, p. 163):

$$[\{d^2 + \nu r^{-1}d + \nu r^{-2}d'^2\} w_2]_{r=c} = 0, \tag{5a}$$

$$[\{d^3 + r^{-1}d^2 - r^{-2}d + (2-\nu)r^{-2}dd'^2 + (\nu-3)r^{-3}d'^2\} w_2]_{r=c} = 0, \tag{5b}$$

where ν is Poisson's ratio for the material of the plate,

(iv) the appropriate singular behaviour near the support points and the vanishing of the deflection at these points.

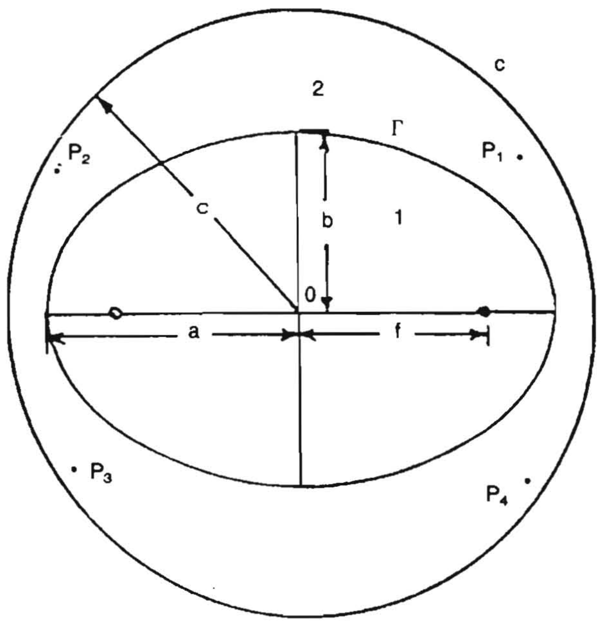

Fig. 1.

Moments and shears at any point can then be computed by applying standard formulae either in polar coordinates (r,θ) or in complex variables $(z,\bar{z})$. (Timoshenko and Woinowsky-Krieger 1959, p. 283).

3. Method and Solution

The general solutions of (3) are

$$w_n = 2 \text{ Re } [\bar{z}\Omega_n(z) + \omega_n(z)] + W_n(z,\bar{z}) \ (n = 1,2), \tag{6}$$

where

$$W_1(z,\bar{z}) = p_0 z^3 \bar{z}^3 / 576D, \ W_2(z,\bar{z}) = 0, \tag{7}$$

and $\Omega_n(z)$, $\omega_n(z)$ are four functions of z which are analytic in their domains. The reaction, measured positively upwards, at any point of support equals L/4 where L is the total load which is given by

$$L = \frac{1}{4} \pi p_0 abg^2, \ g^2 = a^2 + b^2. \tag{8}$$

Symmetry considerations show that it is sufficient to take z in the positive quadrant of the plate. Following the same procedure of Bassali and Nassif (1959), p. 104 and Bassali (1959), p. 112 it can be shown that the transition conditions (4) along Γ are satisfied by

$$2k[\Omega(z)]_1^2 = \frac{abz}{g^2}\left(1 - \frac{4}{3}\xi^2 + \frac{2g^4 - f^4}{10\, a^2 b^2}\xi^4\right) + z \ln \frac{z+Z}{a+b} - \frac{1}{15}(16 - 7\xi^2 + 6\xi^4)Z, \tag{9a}$$

$$2k[\omega(z)]_1^2 = 2ab\left(\frac{1}{4} + \frac{2}{3}\xi^4 + \frac{3f^4 - 4g^4}{45 a^2 b^2}\xi^6\right) + \sigma f^2 \ln \frac{z+Z}{a+b} + z(\alpha_0\xi^4 - \alpha_1\xi^2 - \alpha_2)Z, \tag{9b}$$

where

$$k = \frac{8\pi D}{L}, \; \xi = \frac{z}{f}, \; f^2 = a^2-b^2, \; Z = \sqrt{z^2-f^2}, \; \sigma = \frac{2\lambda+\lambda^{-1}}{6}, \qquad (9c)$$

$$\alpha_0 = \frac{4}{45}(4\lambda-\lambda^{-1}), \; \alpha_1 = \frac{22\lambda-13\lambda^{-1}}{45}, \; \alpha_2 = \frac{6\lambda+11\lambda^{-1}}{30}, \; \lambda = \frac{g^2}{f^2}, \qquad (9d)$$

and the branch of Z taken is that which is positive when z is real and $z^2 > f^2$. The singular parts of the complex potentials $\Omega_n(z)$ and $\omega_n(z)$ near point forces are provided by equations (2.18), p. 732 of Bassali (1957) or by equations (25), p. 269 of Bassali (1958). Taking this into consideration we assume that

$$2k\Omega_1 = c \sum_0^\infty C_n(z/c)^{2n+1} - \frac{1}{4}\sum_1^4 Z_n \ln(Z_n/c)$$

$$- \frac{abz}{g^2}\left(1 - \frac{4}{3}\xi^2 + \frac{2g^4 - f^4}{10\, a^2 b^2}\xi^4\right), \qquad (10a)$$

$$2k\omega_1 = c^2 \sum_0^\infty A_n(z/c)^{2n} + \frac{1}{4}\sum_1^4 \bar{z}_n Z_n \ln(Z_n/c)$$

$$-2ab\left(\frac{1}{4} + \frac{2}{3}\xi^4 + \frac{3f^4-4g^4}{45\, a^2 b^2}\xi^6\right), \qquad (10b)$$

$$2k\Omega_2 = c \sum_0^\infty C_n(z/c)^{2n+1} - \frac{1}{4}\sum_1^4 Z_n \ln(Z_n/c)$$

$$+ z \ln\frac{z+Z}{a+b} - \frac{1}{15}(16 - 7\xi^2 + 6\xi^4)Z, \qquad (11a)$$

$$2k\omega_2 = c^2 \sum_0^\infty A_n(z/c)^{2n} + \frac{1}{4}\sum_1^4 \bar{z}_n Z_n \ln(Z_n/c)$$

$$+ \sigma f^2 \ln\frac{z+Z}{a+b} + z(\alpha_0\xi^4 - \alpha_1\xi^2 - \alpha_2)Z, \qquad (11b)$$

where, to ensure the uniformity of the complex potentials, the terms containing Z appear in Ω_2, ω_2, $Z_n = z - z_n$ (n = 1,2,3,4) and A_n, C_n (n = 0,1,2,...) are real dimensionless constants to be determined from the conditions (5a,b) and the vanishing of the deflection at the support points. Substitution from (10a,b), (11a,b) and (7) in (6) leads to

$$\frac{kw_1}{c^2} = \sum_0^\infty (A_n+C_n\rho^2)\rho^{2n}\cos 2n\theta - S + t_1t_2\left[\frac{4}{3}(\lambda^{-1}\cos 2\theta - \cos 4\theta)\frac{\rho^4}{t^4} - \frac{\rho^2}{v^2} - \frac{1}{2}\right]$$

$$+ \frac{\rho^6}{90t_1t_2v^2}[5+9(1-2\lambda^2)\cos 4\theta + 4\lambda(4\lambda^2-3)\cos 6\theta], \qquad (12a)$$

$$\frac{kw_2}{c^2} = \sum_0^\infty (A_n+C_n\rho^2)\rho^{2n}\cos 2n\theta - S + (\rho^2+\sigma t^2)\ 1n\frac{|z+Z|}{a+b}$$

$$+ \frac{1}{c^2}\mathrm{Re}\left[z(\alpha_0\xi^4 - \alpha_1\xi^2 - \alpha_2) - \frac{1}{15}\bar{z}(16-7\xi^2+6\xi^4)\right]Z, \qquad (12b)$$

where

$$\rho = r/c,\ t = f/c,\ t_1 = a/c,\ t_2 = b/c,\ v = g/c,\ \lambda = v^2/t^2 \qquad (13)$$

and

$$S = \frac{1}{4}\sum_1^4 (R_n^2/c^2)\ 1n(R_n/c),\ R_n = |Z_n|). \qquad (14)$$

It is easily shown that

$$S = (\rho^2+u^2)1n\ \rho + u^2 + \sum_2^\infty \frac{1}{2n}\left(\frac{\rho^2}{2n-1} - \frac{u^2}{2n+1}\right)\left(\frac{u}{\rho}\right)^{2n}\cos 2n\gamma\cos 2n\theta \qquad (15)$$

if $\rho \geq u$ and we interchange ρ and u if $\rho \leq u$. After extensive algebraic manipulation it is found that the expression involving Z in (12b) has the following expansions in terms of biharmonic functions of (ρ,θ):

$$(\rho^2+\frac{1}{3}v^2 + \frac{1}{6}q^2)1n\frac{2\rho}{t_1+t_2} + \frac{5}{36}v^2+\frac{11}{72}q^2-\frac{5}{4}\rho^2+(\frac{2}{3}\rho^2-\frac{1}{2}q^2)\frac{\rho^2}{t^2}\cos 2\theta$$

$$+ \left(\frac{1}{3} q^2 - \frac{2}{3} v^2 - \frac{2}{5} \rho^2\right) \frac{\rho^4}{t^4} \cos 4\theta + \frac{4}{45} (4v^2 - q^2) \frac{\rho^6}{t^6} \cos 6\theta$$

$$+ \sum_1^{\infty} \frac{(t/\rho)^{2n} \delta_n}{n(n+1)(n+2)} \left[\frac{2n+1}{2n-1} \rho^2 - \frac{2(n+1)v^2 + q^2}{2(n+3)}\right] \cos 2n\theta \ (\rho \geqslant t), \tag{16a}$$

$$(\rho^2 + \frac{1}{3} v^2 + \frac{1}{6} q^2) 1n \frac{t}{t_1 + t_2} + \frac{1}{6} (2.8 \rho^2 + 2v^2 + q^2) \frac{\rho}{t} \sin \theta$$

$$+ \frac{1}{45} (12\rho^2 - 2v^2 - 7q^2) \frac{\rho^3}{t^3} \sin 3\theta + \frac{4}{45} (4v^2 - q^2) \frac{\rho^5}{t^5} \sin 5\theta$$

$$+ 8 \sum_2^{\infty} \frac{(\rho/t)^{2n+1} \delta_n}{(4n^2-1)(2n-3)} \left[\frac{(2n-1)v^2 - q^2}{2n-5} - \frac{n\rho^2}{n+1}\right] \sin (2n+1)\theta \ (\rho \leqslant t), \tag{16b}$$

where $q = f^2/cg = t^2/v, \ \delta_n = 2^{-2n} \binom{2n}{n} = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}.$ (17)

Introducing (15) and (16a) in (12b) we obtain the expansion

$$kw_2/c^2 = \sum_0^{\infty} L_n(\rho) \cos 2n\theta, \tag{18}$$

where $\rho \geqslant$ the greater of t and u,

$$L_0(\rho) = A_0' + B_0' \, 1n \, \rho + C_0' \rho^2, \quad L_n(\rho)$$

$$= A_n' \rho^{2n} + B_n' \rho^{-2n} + C_n' \rho^{2+2n} + D_n' \rho^{2-2n} (n \geqslant 1), \tag{19}$$

$$A_0' = A_0 + \frac{5}{36} v^2 + \frac{11}{72} q^2 - u^2 + \frac{1}{6} (2v^2 + q^2) 1n \frac{2}{t_1 + t_2}, \tag{20a}$$

$$B_0' = \frac{1}{3} v^2 + \frac{1}{6} q^2 - u^2, \ C_0' = C_0 - \frac{5}{4} + 1n \frac{2}{t_1 + t_2}, \tag{20b}$$

$$A_1' = A_1 - \frac{t^2}{2v^2}, \quad A_2' = A_2 + \frac{q^2-2v^2}{3t^4}, \quad A_3' = A_3 + \frac{4(4v^2-q^2)}{45t^6}, \tag{20c}$$

$$C_1' = C_1 + \frac{2}{3t^2}, \quad C_2' = C_2 - \frac{2}{5t^4}, \tag{20d}$$

$$A_n' = A_n \ (n \geq 4), \quad B_n' = \frac{1}{2n}\left[\frac{u^{2n+2}}{2n+1}\cos 2n\gamma - \frac{t^{2n}\delta_n\beta_n}{n+2}\right] (n \geq 1), \tag{20e}$$

$$C_n' = C_n \ (n \geq 3), \quad D_n' = \frac{1}{2n(2n-1)}\left[\frac{2(2n+1)t^{2n}\delta_n}{(n+1)\,(n+2)} - u^{2n}\cos 2n\gamma\right] (n \geq 1), \tag{20f}$$

where
$$\beta_n = \frac{1}{n+3}\left(2v^2 + \frac{q^2}{n+1}\right) \tag{21}$$

Inserting (18) in (5a,b), equating the coefficients of $\cos 2n\theta$ $(n=0,1,2,\ldots)$ in the two identities to zero and solving the resulting systems of linear equations we get

$$C_0' = \frac{1}{2}\beta\left(\frac{1}{3}v^2 + \frac{1}{6}q^2 - u^2\right), \tag{22a}$$

$$A_n' = \frac{1}{2n\kappa}\left[u^{2n}(u^2-\alpha_n)\cos 2n\gamma + \frac{2n+1}{n+2}t^{2n}\delta_n\left(\frac{2\alpha_n}{n+1} - \beta_n\right)\right] (n \geq 1), \tag{22b}$$

$$C_n' = \frac{1}{\kappa}\left[u^{2n}\left(\frac{1}{2n} - \frac{u^2}{2n+1}\right)\cos 2n\gamma + \frac{t^{2n}\delta_n}{n+2}\left(\beta_n - \frac{1}{n} - \frac{1}{n+1}\right)\right] (n \geq 1), \tag{22c}$$

where
$$\beta = \frac{1-\nu}{1+\nu}, \quad \kappa = \frac{3+\nu}{1-\nu}, \quad \alpha_n = \frac{4n^2+\kappa^2-1}{2n(2n-1)}. \tag{23}$$

The values of the constants A_n and C_n $(n=0,1,2,\ldots)$ are thus completely determined except A_0. Substituting for A_1, A_2, A_3, C_1, C_2 from (20 c,d) in (12 a,b) and introducing the notation

$$m = (a-b) / (a+b) , \tag{24}$$

we find

$$\frac{kw_1}{c^2} = A_0+C_0\rho^2 + \sum_1^\infty (A_n'+C_n' \rho^2) \rho^{2n} \cos 2n\theta - S - t_1t_2\left(\frac{\rho^2}{v^2}+\frac{1}{2}\right)$$

$$+ \frac{\rho^2}{v^2}\left\{\left(\frac{1}{2}t^2-\frac{2}{3}m\rho^2\right)\cos 2\theta+m^2\rho^2\left(\frac{1}{3}-\frac{\rho^2}{10t_1t_2}\right)\cos 4\theta+\frac{\rho^4}{90t_1t_2}(5+4m^3\cos 6\theta)\right\}, \tag{25a}$$

$$\frac{kw_2}{c^2} = A_0+C_0\rho^2 + \sum_1^\infty (A_n'+C_n' \rho^2) \rho^{2n} \cos 2n\theta - S + (\rho^2+\frac{1}{3}v^2+\frac{1}{6}q^2)\ 1n\frac{|z+Z|}{a+b}$$

$$+ (\frac{1}{2}q^2-\frac{2}{3}\rho^2)\frac{\rho^2}{t^2}\cos 4\theta + (\frac{2}{5}\rho^2+\frac{2}{3}v^2-\frac{1}{3}q^2)\frac{\rho^4}{t^4}\cos 4\theta + \frac{4}{45}(q^2-4v^2)\frac{\rho^6}{t^6}\cos 6\theta$$

$$+ \frac{1}{c^2}\,\mathrm{Re}\left[(\alpha_0z^2-\frac{2}{5}r^2)\frac{z^4}{f^4}+(\frac{7}{15}r^2-\alpha_1z^2)\frac{z^2}{f^2}-(\frac{16}{15}r^2+\alpha_2z^2)\right]\sqrt{(1-\frac{f^2}{z^2})}. \tag{25b}$$

The expression containing the square root in (25b) has the expansion (16a) if $r \geq f$ ($\rho \geq t$) and the expansion (16b) if $r \leq f$ ($\rho \leq t$). Substituting from (16a) in (25b) yields

$$\frac{kw_2}{c^2} = A_0+C_0\rho^2 + \sum_1^\infty (A_n'+C_n' \rho^2) \rho^{2n} \cos 2n\theta - S + \frac{5}{36}v^2$$

$$+ \frac{11}{72}q^2-\frac{5}{4}\rho^2 + (\rho^2+\frac{1}{3}v^2+\frac{1}{6}q^2)\ 1n\frac{2\rho}{t_1+t_2}$$

$$+ \sum_1^\infty \frac{(t/\rho)^{2n}\delta_n}{n(n+1)(n+2)}\left[\frac{2n+1}{2n-1}\rho^2 - \frac{2(n+1)v^2+q^2}{2(n+3)}\right]\cos 2n\theta\ (\rho \geq t). \tag{25c}$$

Introducing (16b) in (25b) gives the expansion of kw_2/c^2 at points of region 2 where $r \leq f$. It is easily seen that such points exist only if the eccentricity of the ellipse $\geq 1/\sqrt{2}$. Whether $r \geq f$ or $r \leq f$ the deflection w_2 is furnished by (12b) where

$$|z+Z| = [r^2+T^2+ \sqrt{2}\, r \sqrt{(T^2 + r^2 - f^2 \cos 2\theta)}]^{\frac{1}{2}}, \tag{26a}$$

$$\mathrm{Re}(z^n Z) = \frac{r^n}{\sqrt{2}} [\cos n\theta \sqrt{(T^2+r^2 \cos 2\theta - f^2)} - \sin n\theta \sqrt{(T^2+f^2-r^2 \cos 2\theta)}], \tag{26b}$$

$$n = 0, \pm 1, \pm 2, \ldots, T^4 = r^4 + f^4 - 2f^2 r^2 \cos 2\theta. \tag{26c}$$

Setting $b = a$ ($t_2=t_1$), $m=0$ in (25a) and evaluating the limit of (25b) or (25c) as $f \to 0$ we arrive at the following equations for the deflection surface of the circular plate corresponding to normal loading of intensity $p_0 r^2$ over a concentric circle of radius a and four supports at the corners of a concentric rectangle:

$$\frac{kw_1}{c^2} = A_0 - \frac{1}{2} t_1^2 + \left[\frac{3}{4} + 1n\ t_1 + \beta\left(\frac{1}{3} t_1^2 - \frac{1}{2} u^2\right) + \frac{\rho^4}{36 t_1^4}\right] \rho^2 - S$$

$$+ \frac{1}{\kappa} \sum_1^\infty \left[\left(\frac{1}{2n} - \frac{u^2}{2n+1}\right) \rho^2 + \frac{1}{2n}\left\{u^2 - 1 - \frac{1}{2n} - \frac{\kappa^2}{2n(2n-1)}\right\}\right] \times$$

$$(\rho u)^{2n} \cos 2n\gamma \cos 2n\theta, \tag{27a}$$

$$\frac{kw_2}{c^2} = A_0 + [1n\ \rho + \beta(\frac{1}{3} t_1^2 - \frac{1}{2} u^2)] + \frac{1}{3}\left(\frac{5}{6} + 2\ 1n \frac{\rho}{t_1}\right) t_1^2 - S$$

$$+ \frac{1}{\kappa} \sum_1^\infty \left[\left(\frac{1}{2n} - \frac{u^2}{2n+1}\right) \rho^2 + \frac{1}{2n}\left\{u^2 - 1 - \frac{1}{2n} - \frac{\kappa^2}{2n(2n-1)}\right\}\right] \times$$

$$(\rho u)^{2n} \cos 2n\gamma \cos 2n\theta, \tag{27b}$$

It is worthy of mentioning here that the general problems of a thin circular plate supported at several interior or boundary points and acted upon by two types of normal loadings over the area of an eccentric circle were studied by one of the authors (Bassali 1957, 1958). Letting t_1 tend to zero in (27b) yields the solution corresponding to the four supports and a concentrated central load L. It is verified that equations (27a,b) agree with the expressions obtained by putting $m=4$, $n=4$, $n'=6$, $P_1=P_2=P_3=P_4 = -L/4$, $P_0=L$ in equations (40a,b), p. 735 of Bassali (1957) on noting the difference in notation and making some expansions of the logarithmic functions.

We have now to determine the remaining constant A_0. This depends on the positions of the support points which lie either in region 1 or in region 2 according as $s^2 \cos^2\gamma/a^2 + s^2 \sin^2\gamma/b^2 \leqslant 1$ or $\geqslant 1$. In the first case A_0 is found by equating to zero the expression obtained by putting $\theta=\gamma$ and $\rho=u$ in (25a). In the second case either $s \geqslant f$ or $s \leqslant f$. If $s \geqslant f$ equation (18) can be used to determine A_0' and then A_0 is given by (20a). Whether $s \geqslant f$ or $s \leqslant f$ equations (25b) and (26a,b,c) can be used to determine A_0 but the expansions (16a) or (16b) may be applied according as $s \geqslant f$ or $s \leqslant f$, respectively. In any case the deflection at the centre is given by

$$W_0 = \frac{Lc^2}{8\pi D} (A_0 - \frac{1}{2} t_1 t_2 - u^2 \ln u). \tag{28}$$

4. Moments and Shears

Substituting from (7), (10a,b) and (11a,b) in the standard formulae given by equations (1.6), p. 730 of Bassali (1957) we obtain the following expressions for the bending, twisting moments and shearing forces at any point z of the plate:

$$\begin{array}{l} M_r^1 \\ M_\theta^1 \end{array} = \frac{(1+\nu)L}{4\pi} \Bigg[1 + \frac{1}{8} \ln(v_1 v_2) \pm \frac{1}{8} \beta (1 - \frac{u^2}{\rho^2}) \{2 + (\rho^4 - u^4)(v_1^{-1} + v_2^{-1})\}$$

$$- C_0 - \sum_1^\infty \left\{ (2n+1)(1 \pm n\beta) C_n \pm \frac{\beta n(2n-1)}{\rho^2} A_n \right\} \rho^{2n} \cos 2n\theta$$

$$+ \frac{t_1 t_2}{v^2} - \frac{\rho^4}{t_1 t_2 v^2} (\frac{1}{2} \pm \frac{1}{3}\beta) - 4(1 \pm \beta) \frac{\rho^2 t_1 t_2}{t^2 v^2} \cos 2\theta + \rho^2 \Bigg\{ \frac{\pm 8\beta t_1 t_2}{t^4}$$

$$+ \frac{(2\lambda^2 - 1)\rho^2}{t_1 t_2 v^2} (\frac{1}{2} \pm \beta) \Bigg\} \cos 4\theta \pm \frac{2\beta(3 - 4\lambda^2)}{3 t_1 t_2 t^2} \rho^4 \cos 6\theta \Bigg], \tag{29a,b}$$

$$M_{r\theta}^1 = \frac{(1-\nu)L}{4\pi} \left[\frac{1}{4} u^2 (u^2 - \rho^2) \left(\frac{\sin 2\phi_1}{v_1} + \frac{\sin 2\phi_2}{v_2} \right) \right.$$

$$- \sum_{1}^{\infty} n \left\{(2n+1)\, C_n + \frac{2n-1}{\rho^2} A_n\right\} \rho^{2n} \sin 2n\theta$$

$$+ \frac{\rho^2}{v^2} \left(\frac{2\lambda^2-1}{t_1 t_2} \rho^2 \sin 4\theta - \frac{4t_1t_2}{t^2} \sin 2\theta\right)$$

$$+ \frac{2\rho^2}{t^2} \left(\frac{4t_1t_2}{t^2} \sin 4\theta + \frac{3-4\lambda^2}{3t_1t_2} \rho^2 \sin 6\theta\right)\Bigg], \tag{29c}$$

$$Q_r^1 = \frac{L}{\pi r} \left[\frac{1}{4} + \frac{1}{8} (\rho^4 - u^4)\, (v_1^{-1} + v_2^{-1})\right.$$

$$- \sum_{1}^{\infty} n(2n+1)\, C_n\, \rho^{2n} \cos 2n\theta$$

$$\left. + \frac{\rho^2}{v^2} \left(\frac{2\lambda^2-1}{t_1t_2} \rho^2 \cos 4\theta - \frac{4t_1t_2 \cos 2\theta}{t^2}\right) - \frac{\rho^4}{t_1t_2v^2}\right], \tag{30a}$$

$$Q_\theta^1 = \frac{L}{\pi r} \left[\frac{1}{4} \rho^2 u^2 \left(\frac{\sin 2\phi_1}{v_1} + \frac{\sin 2\phi_2}{v_2}\right)\right.$$

$$+ \sum_{1}^{\infty} n(2n+1)\, C_n\, \rho^{2n} \sin 2n\theta$$

$$\left. - \frac{\rho^2}{v^2} \left(\frac{2\lambda^2-1}{t_1t_2} \rho^2 \sin 4\theta - \frac{4t_1t_2 \sin 2\theta}{t^2}\right)\right]; \tag{30b}$$

$$\begin{matrix} M_r^2 \\ M_\theta^2 \end{matrix} = \frac{(1+\nu)L}{4\pi} \left[1 + \frac{1}{8} \ln(v_1v_2) \pm \frac{1}{8} \beta\left(1 - \frac{u^2}{\rho^2}\right) \{2 + (\rho^4 - u^4)\, (v_1^{-1} + v_2^{-1})\}\right.$$

$$-C_0 - \sum_{1}^{\infty} \left\{(2n+1)\, (1 \pm n\beta) C_n \pm \frac{\beta n(2n-1)}{\rho^2} A_n\right\} \rho^{2n} \cos 2n\theta - \ln \frac{|z+Z|}{a+b}$$

$$+ \operatorname{Re} \frac{zZ}{f^2} \left\{ -1 + \frac{2z^2}{f^2} \pm 4\beta \left(\frac{1}{f^2} + \frac{\lambda - \lambda^{-1}}{3r^2} \right) z^2 \pm \frac{4\beta z^4}{3r^2 f^2} (\lambda^{-1} - 4\lambda) \right\} \Bigg] , \quad (31a,b)$$

$$M^2_{r\theta} = \frac{(1-\nu)L}{4\pi} \left[\frac{1}{4} u^2(u^2-\rho^2) \left(\frac{\sin 2\,\phi_1}{v_1} + \frac{\sin 2\,\phi_2}{v_2} \right) \right.$$

$$- \sum_1^\infty n \left\{ (2n+1)\, C_n + \frac{2n-1}{\rho^2} A_n \right\} \rho^{2n} \sin 2n\theta$$

$$\left. + 4 \operatorname{Im} \frac{z^3 Z}{f^4} \left\{ 1 + (\lambda - \lambda^{-1}) \frac{f^2}{3r^2} + (\lambda^{-1} - 4\lambda) \frac{z^2}{3r^2} \right\} \right] , \quad (31c)$$

$$Q^2_r = \frac{L}{\pi r} \left[\frac{1}{4} + \frac{1}{8} (\rho^4 - u^4)(v_1^{-1} + v_2^{-1}) + 4 \operatorname{Re} (z^3 Z/f^4) \right.$$

$$\left. - \sum_1^\infty n(2n+1)\, C_n\, \rho^{2n} \cos 2n\theta \right] , \quad (32a)$$

$$Q^2_\theta = \frac{L}{\pi r} \left[\frac{1}{4} \rho^2 u^2 \left(\frac{\sin 2\phi_1}{v_1} + \frac{\sin 2\phi_2}{v_2} \right) - 4 \operatorname{Im} (z^3 Z/f^4) \right.$$

$$\left. + \sum_1^\infty n(2n+1)\, C_n\, \rho^{2n} \sin 2n\theta \right] , \quad (32b)$$

where $\phi_1 = \theta - \gamma\,, \quad \phi_2 = \theta + \gamma\,,$ (33)

$$v_1 = \rho^4 + u^4 - 2u^2\rho^2 \cos 2\phi_1, \quad v_2 = \rho^4 + u^4 - 2u^2\rho^2 \cos 2\phi_2, \quad (34)$$

$|z+Z|$, $\operatorname{Re}(z^n Z)$ are given by (26a,b,c) and

$$\operatorname{Im}(z^n Z) = \frac{r^n}{\sqrt{2}} [\cos n\theta \sqrt{(T^2 - r^2 \cos 2\theta + f^2)} + \sin n\theta \sqrt{(T^2 + r^2 \cos 2\theta - f^2)}]. \quad (35)$$

At the centre of the plate we have $(Q_r)_0 = (Q_\theta)_0 = 0$,

$$\begin{matrix}(M_r)_0\\(M_\theta)_0\end{matrix} = \frac{(1+\nu)L}{4\pi}\left[\frac{t_1t_2}{v^2} - \frac{1}{4} + \ln\frac{2u}{t_1+t_2} + \frac{1}{2}\beta(u^2-\frac{1}{3}v^2-\frac{1}{6}q^2)\right.$$

$$\left.\pm\frac{1}{2}\beta\cos 2\theta\left\{\cos 2\gamma - \frac{t^2}{v^2} + \frac{u^2}{2\kappa}(\kappa^2+3-2u^2) + \frac{t^2}{4\kappa}(v^2+\frac{1}{4}q^2-\kappa^2-3)\right\}\right], \tag{36a,b}$$

$$(M_{r\theta})_0 = \frac{(1-\nu)L}{8\pi}\left[\cos 2\gamma - \frac{t^2}{v^2} + \frac{u^2}{2\kappa}(\kappa^2+3-2u^2)\right.$$

$$\left.+\frac{t^2}{4\kappa}(v^2+\frac{1}{4}q^2-\kappa^2-3)\right]\sin 2\theta. \tag{36c}$$

5. Limiting Cases

Results concerning the deflections, moments and shears in the three following limiting cases can be derived from the foregoing formulae:

(i) Allowing c to tend to ∞ yields the appropriate solutions for an infinitely large plate subject to paraboloidal loading over the area of an ellipse and supported at the four corners of a rectangle concentric with the ellipse and having its sides parallel to the axes of the ellipse.

(ii) The case in which $b\to a$, $f\to 0$ leads to the problem of a circular plate acted upon by the loading $p=p_0r^2$ over a concentric circle and supported at the four vertices of a concentric rectangle. In this case the deflections are given by (27a,b) where A_0 is determined by equating to zero the expressions obtained by setting $\rho=u$ and $\theta=\gamma$ in (27a) or (27b) according as the point supports lie in the loaded or unloaded region, respectively. Moments and shears at any point can either be found by limiting processes from equations (20)−(36) or as special cases of the general formulae (2.44)−(2.46), p. 736 of Bassali (1957).

(iii) In the limiting case in which the minor axis of the ellipse $\to 0$ the loaded patch reduces to a line loading extending along the x-axis from $x=-a$ to $x=a$. Assuming that $b\to 0$ and $p_0\to\infty$ such that $2bp_o\to p_1$ we see that the intensity p of this line loading at a point distant x from the centre is given by

$$p = p_1 x^2\sqrt{1-x^2/a^2}. \tag{37}$$

Deflections, moments and shears corresponding to this line loading along the x-axis and four supports at the corners of a concentric rectangle whose sides are parallel to the coordinate axes are derived by putting $b=0$, $f=g=a$ and $L=\frac{1}{8}\pi p_1a^3$ in the established formulae.

6. Numerical Results

In this section we give the numerical and graphical representation of the deflections, moments and shears at various points in the first quadrant of the circular plate corresponding to specified dimensions of the loaded elliptic patch. It is assumed that $t_1 = 0.6$, $t_2 = 0.45$, $\nu=0.3$. Two different distributions of the point supports in the unloaded region are considered. In the first case we take $u=0.8$, $\gamma=30^o$ while in the second case we take $u=1$, $\gamma=30^o$ so that the point supports lie on the boundary of the circular plate.

The deflection w, radial and transverse bending moments M_r, M_θ, twisting moment $M_{r\theta}$, shearing forces Q_r, Q_θ at any point (r,θ) of the first quadrant may be put in the forms

$$w=\alpha p_0 c^6/D, M_r = \beta_1 p_0 c^4,\ M_\theta = \beta_2 p_0 c^4, M_{r\theta} = \beta_3 p_0 c^4, Q_r = \gamma_1 p_0 c^3, Q_\theta = \gamma_2 p_0 c^3,$$

where $\alpha, \beta_1, \beta_2, \beta_3, \gamma_1$ and γ_2 are dimensionless quantities. Numerical values for these coefficients at points on various radii of the first quadrant are listed in Tables 1-8 and graphs showing their variation are plotted in Figs. 2-13 for both cases of point support distributions.

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Table 1. Case of u = .8 , θ = 0

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	.5050	.4769	.4301	.3630	.2832	.1876	.0832	.0239	−.1382	−.2317
β_1	1.2883	1.2714	1.2212	1.1081	.8974	.3363	.2338	.03674	.0032	0
β_2	.7177	.7297	.7425	.7447	.7178	.6312	.5088	.4073	.3337	.2932
β_3	0	0	0	0	0	0	0	0	0	0
γ_1	.0022	.0298	−.1273	−.3165	−.6175	−1.0529	−.6736	−.3763	−.1607	−.0184
γ_2	0	0	0	0	0	0	0	0	0	0

Table 2. Case of u = .8 , θ = 30°

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	.5058	.4801	.4368	.3759	.2982	.2062	.1049	0	−.1003	−.2012
β_1	1.1495	1.1303	1.1292	1.0497	.8565	.3052	.1022	∞	.1238	0
β_2	.8554	.8439	.8100	.7315	.5761	.3099	− .0609	∞	−.3503	−.2794
β_3	.2473	.2380	.2248	.2110	.2013	.1944	.1712	.1623	.1213	.1021
γ_1	−.0031	−.0486	−.1818	−.4535	−.9282	−1.3116	−1.8147	∞	.7841	.0939
γ_2	−.0076	−.0192	−.0391	−.0702	−.1143	−.0775	−.0093	.0174	.0279	.0274

Table 3. Case of u = .8 , $\theta = 60°$

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	.5075	.4869	.4529	.4067	.3510	.2906	.2305	.1742	.1239	.0794
β_1	.8584	.8318	.8128	.7023	.4784	.2570	.1069	.0263	−.0019	0
β_2	1.1444	1.1345	1.1104	1.0601	.9632	.8623	.7688	.6857	.6173	.3672
β_3	.2549	.2670	.2850	.3047	.3193	.3067	.2713	.2313	.2005	.1821
γ_1	−.0109	−.0632	−.1958	−.4437	−.6994	−.5485	−.3895	−.2448	−.1333	−.0370
γ_2	−.0062	−.0082	−.0022	.0143	.0659	.1808	.2448	.2568	.2327	.1933

Table 4. Case of u = .8 , $\theta = 90°$

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	.5083	.4905	.4619	.4447	.3823	.3395	.3008	.2695	.2480	.2351
β_1	.7066	.6828	.6266	.3097	.3155	.1388	.0783	.0282	.0053	0
β_2	1.2953	1.3015	1.2975	1.2633	1.1818	1.0908	1.0042	.9261	.8582	.8009
β_3	0	0	0	0	0	0	0	0	0	0
γ_1	−.0137	−.0627	−.1844	−.4194	.5227	−.3951	−.2897	−.2072	−.1448	−.0989
γ_2	0	0	0	0	0	0	0	0	0	0

Table 5. Case of u = 1. , θ = 0°

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	1.1751	1.1311	1.0579	.9558	.8267	.6743	.5050	.3259	.1422	−.0430
β_1	1.8538	1.8444	1.8056	1.7026	1.4894	1.1106	.6912	.3706	.1411	.0
β_2	.9883	1.0032	1.0198	1.0258	1.0035	.9283	.8337	.7761	.7451	.7261
β_3	0	0	0	0	0	0	0	0	0	0
γ_1	.0164	−.0039	−.0968	−.2965	−.6350	−1.1417	−.8494	−.6101	−.3966	−.2699
γ_2	0	0	0	0	0	0	0	0	0	0

Table 6. Case of u = 1 , θ = 45

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	1.1788	1.1459	1.0909	1.0139	.9165	.8018	.6762	.5394	.4003	.2616
β_1	1.4225	1.4268	1.4082	1.3276	1.1299	.8519	.6041	.3691	.1514	0
β_2	1.4135	1.3955	1.3548	1.2757	1.1371	.9551	.7861	.6519	.5753	.5594
β_3	.4376	.4434	.4555	.4775	.5132	.5509	.5823	.6107	.6161	.5826
γ_1	−.0070	−.0562	−.1896	−.4491	−.8754	−.8433	−.7595	−.6520	−.4622	−.0900
γ_2	−.0225	−.0448	−.0661	−.0838	−.0931	.0508	.1977	.3597	.5121	.6003

Table 7. Case of u = 1. , θ = 60°

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	1.1807	1.1536	1.1088	1.0474	.9719	.8873	.7985	.7101	.6259	.5489
β_1	1.1999	1.1899	1.1466	1.0306	.7953	.5469	.3425	.1817	.0677	.0
β_2	1.6333	1.6212	1.5923	1.5351	1.4317	1.3194	1.2239	1.1463	1.0826	1.0261
β_3	.3829	.3987	.4237	.4559	.4906	.5061	.4991	.4722	.4298	.3819
γ_2	−.0181	−.0768	−.2161	−.4746	−.7511	−.6354	−.5178	−.3993	−.2855	−.2400
γ_1	−.0191	−.0359	−.0483	−.0541	.0253	.0769	.1532	.2035	.2240	.2626

Table 8. Case of u = 1 , θ = 90°

ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.
α	1.1826	1.1614	1.1274	1.0828	1.0311	.9777	.9271	.8830	.8482	.8249
β_1	.9729	.9386	.8627	.7162	.4837	.2942	.1621	.7591	.2503	0
β_2	1.8574	1.8621	1.8564	1.8206	1.7368	1.6411	1.5452	1.4522	1.3640	1.2845
β_3	0	0	0	0	0	0	0	0	0	0
γ_1	−.0287	−.0944	−.2348	−.4889	−.6083	−.4909	−.3888	−.3033	−.2339	−.1257
γ_2	0	0	0	0	0	0	0	0	0	0

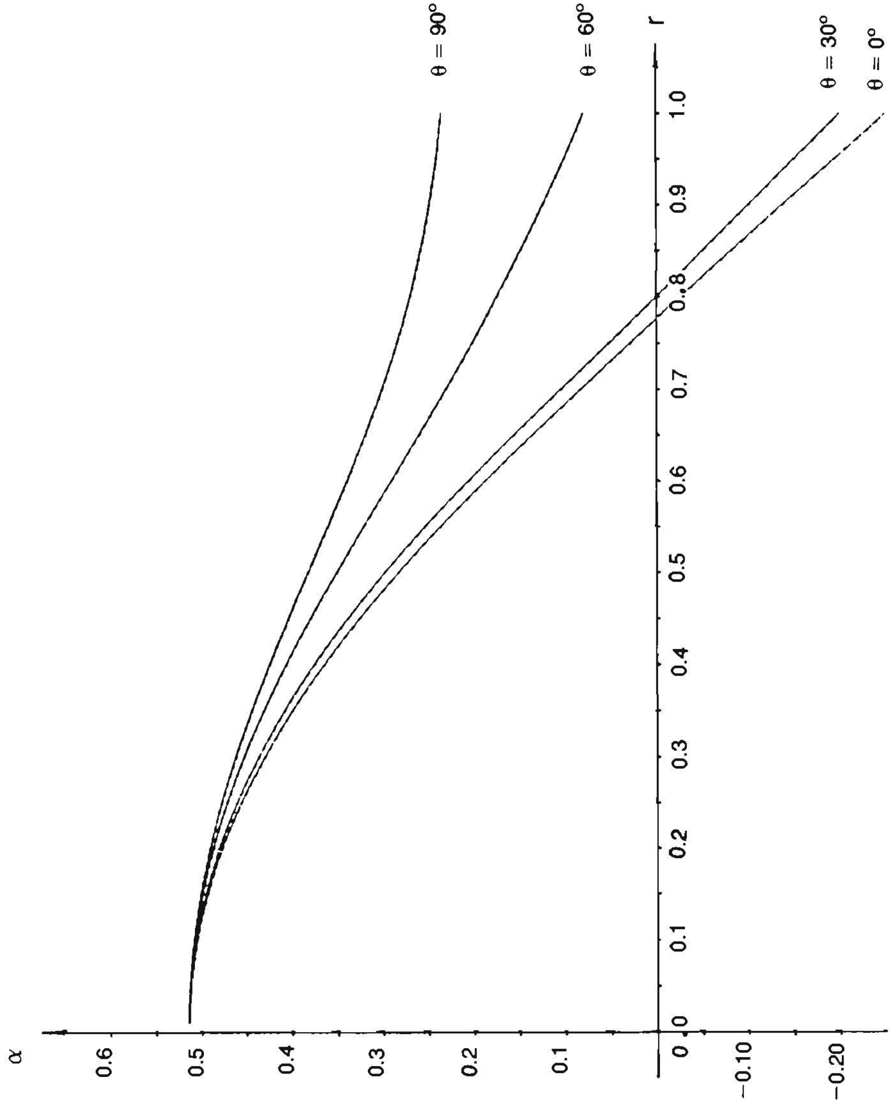

Fig. 2. Deflection profiles along rays, u=.8

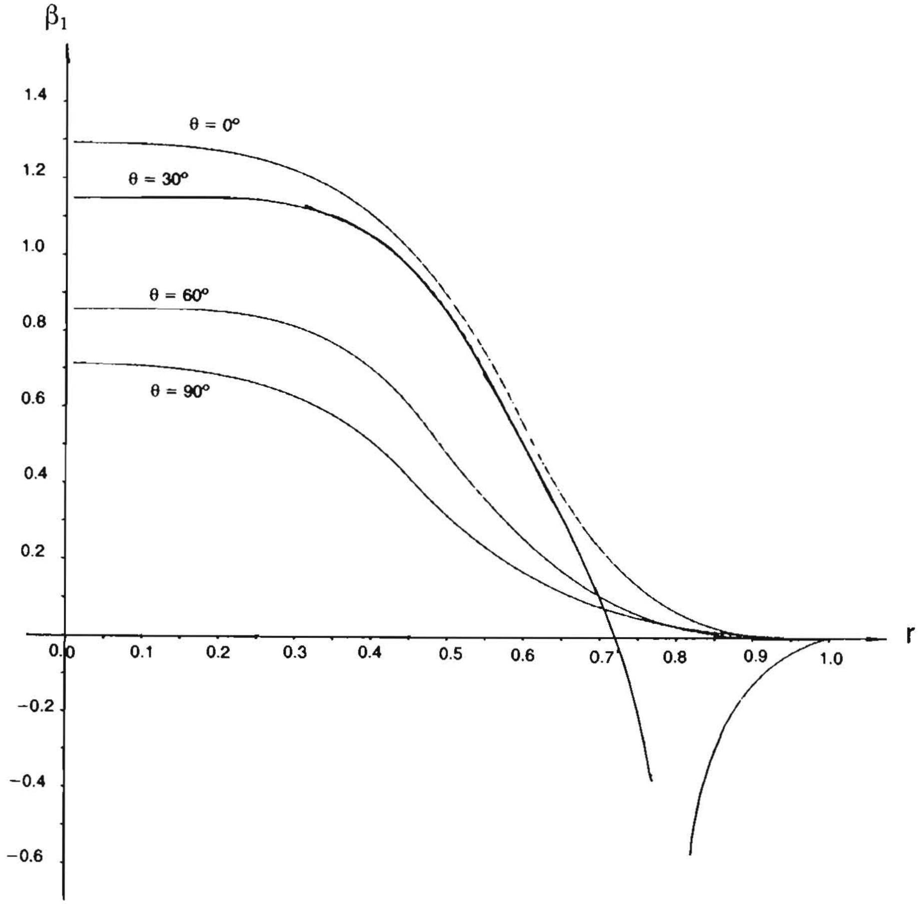

Fig. 3. Radial moment factor profiles along rays, u=.8

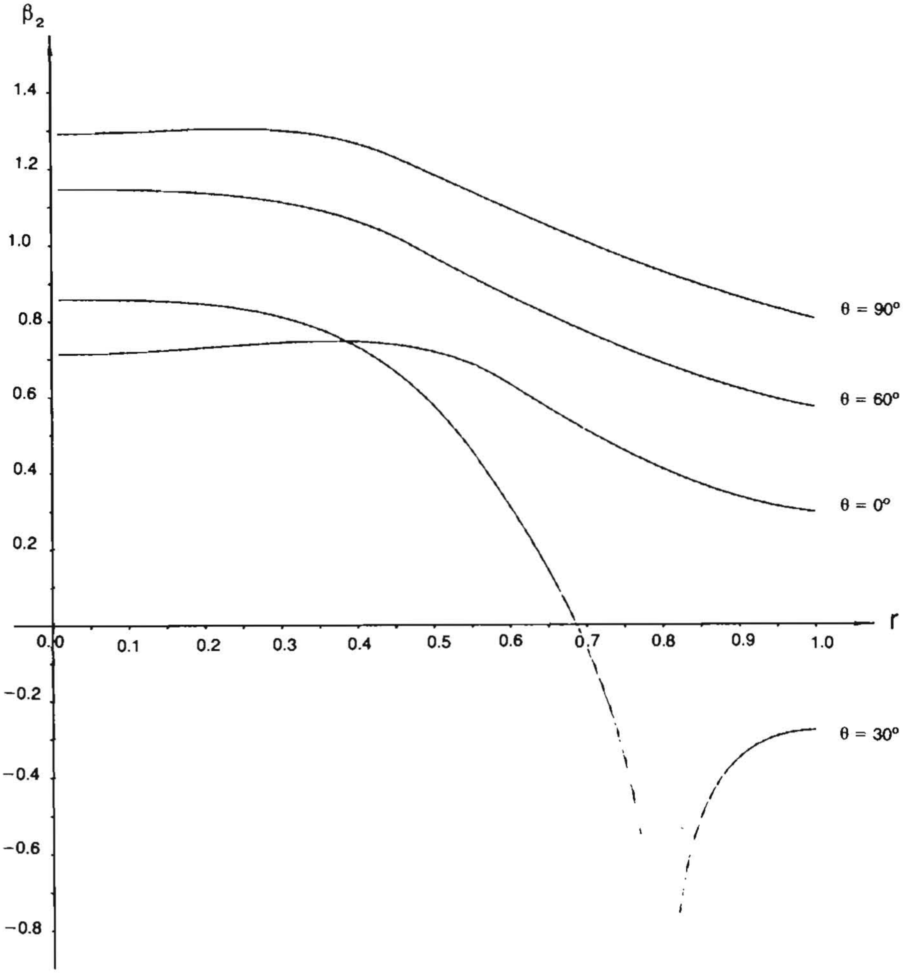

Fig. 4. Transverse moment factor profiles along rays, u=.8

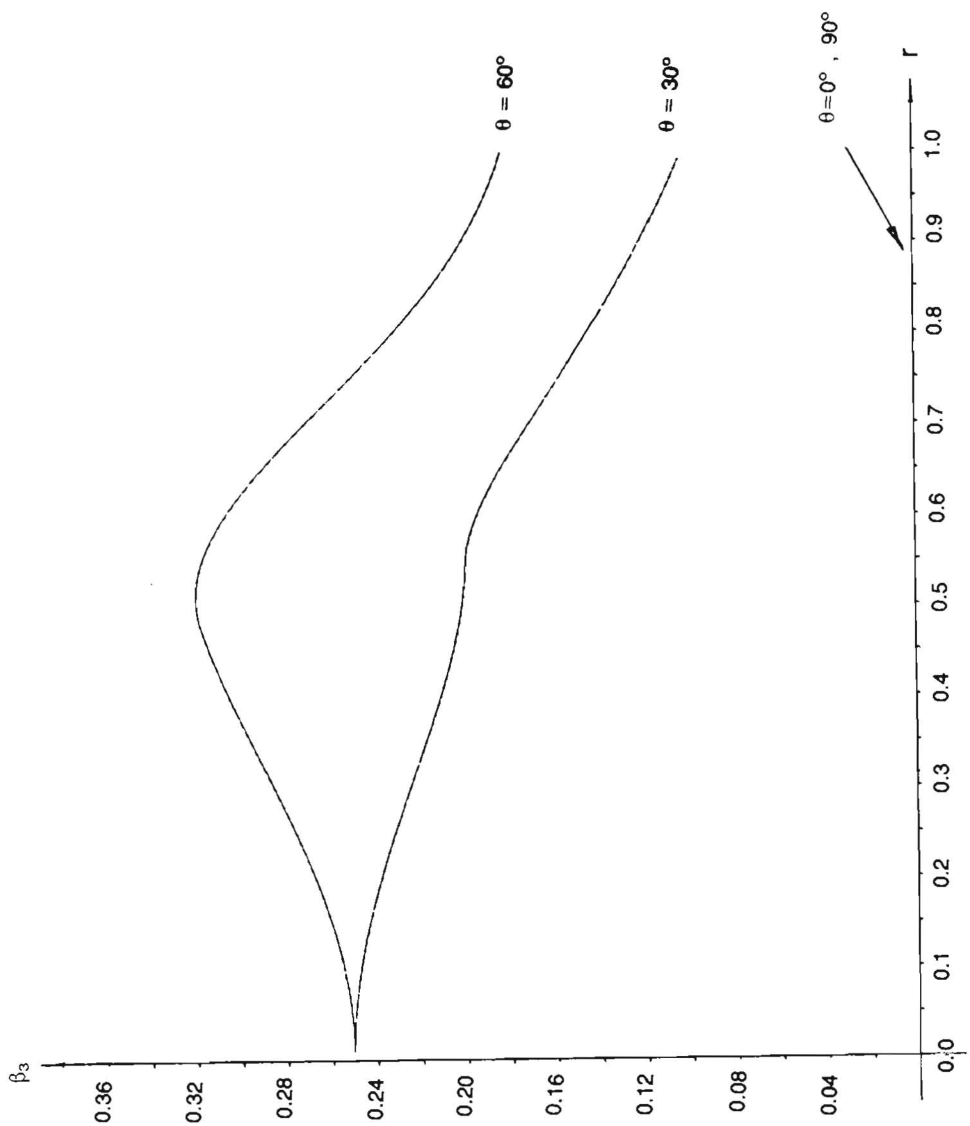

Fig. 5. Twisting moment factor profiles along rays, u=.8

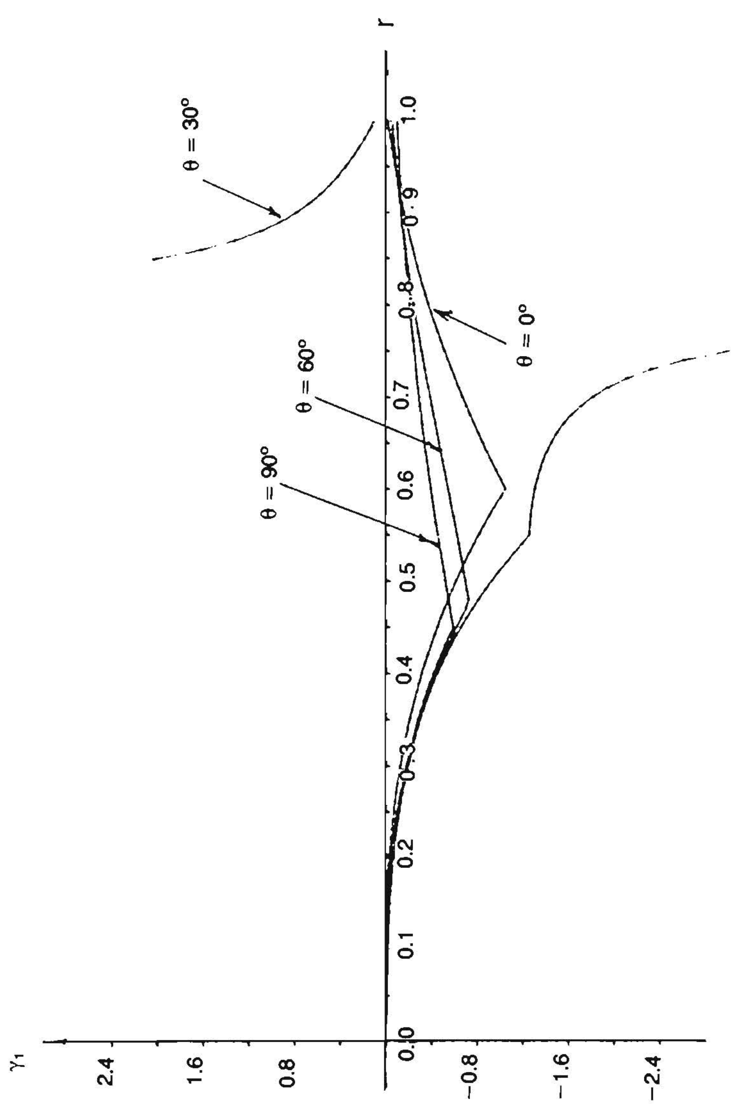

Fig. 6. Radial shear factor profiles along rays, u=.8

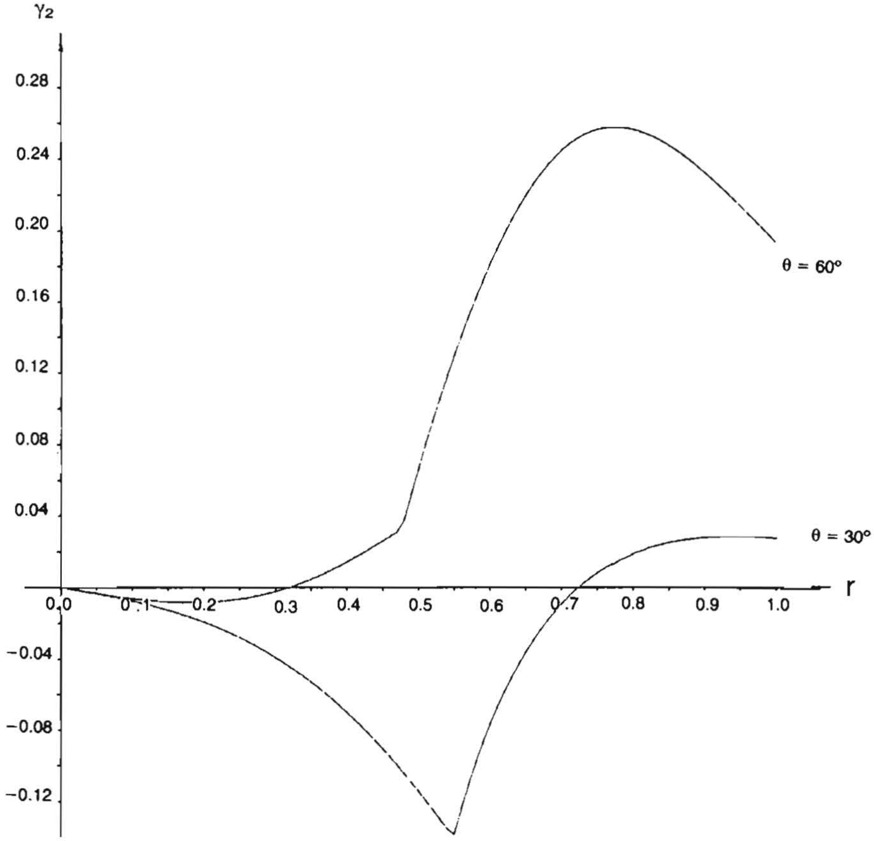

Fig. 7. Transverse shear factor profiles along rays, u=.8

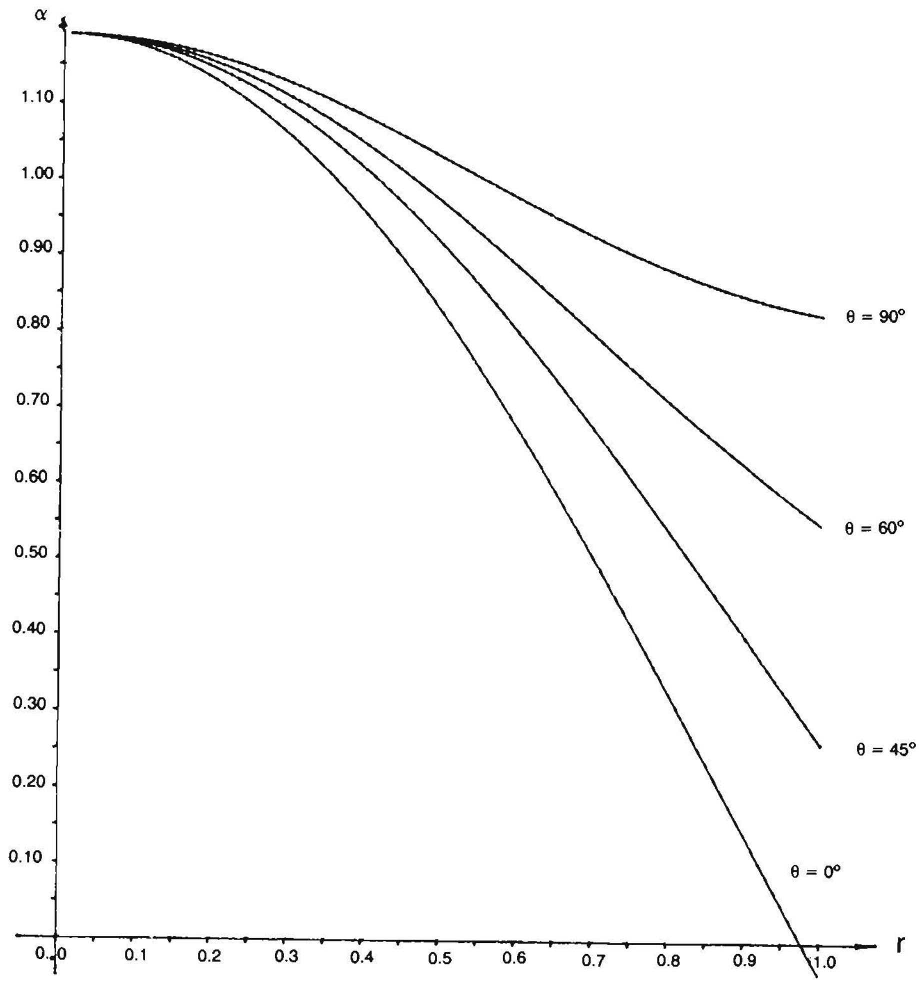

Fig. 8. Deflection profiles along rays, u=1

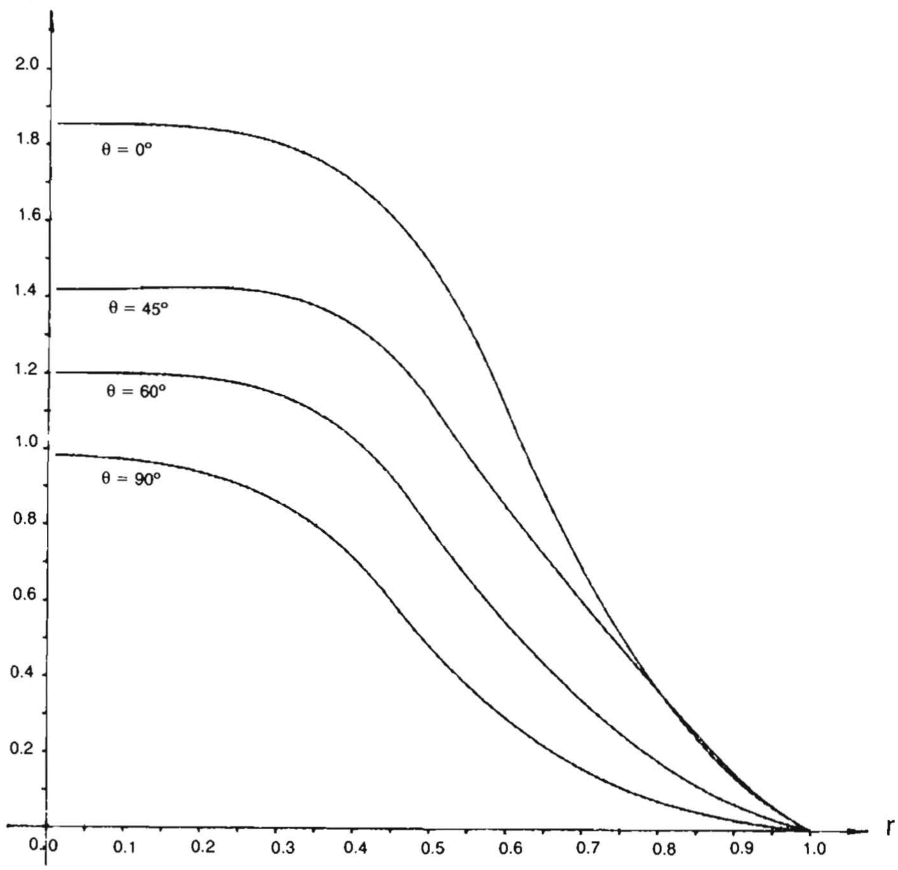

Fig. 9. Radial moment profiles along rays, u=1

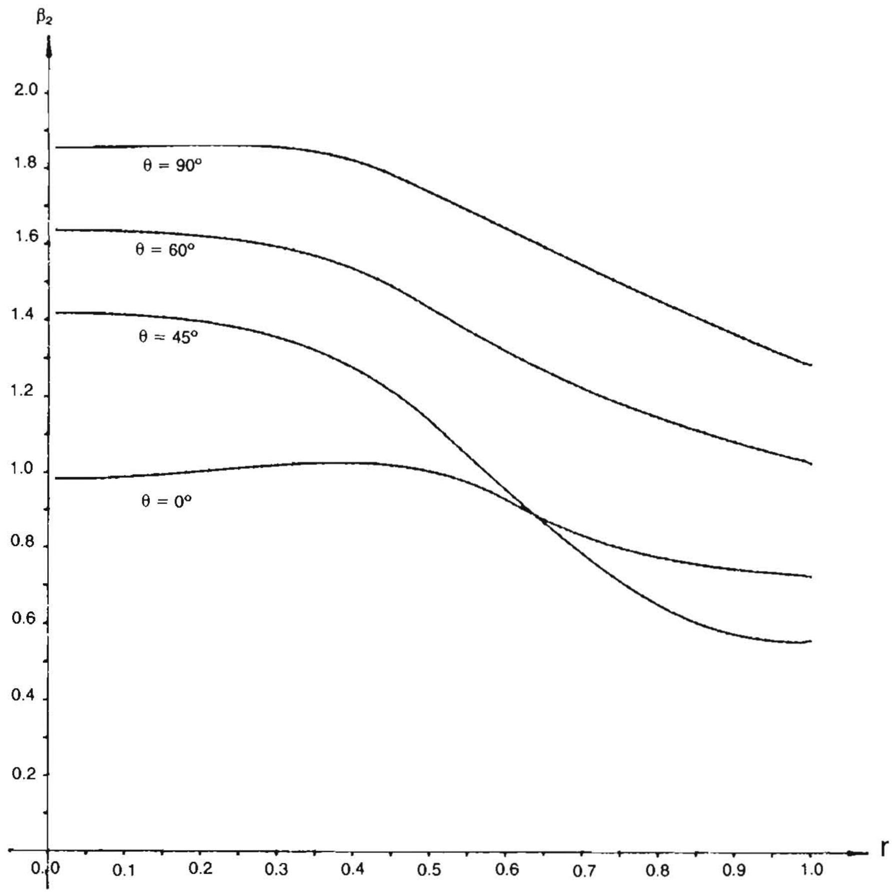

Fig. 10. Transverse moment profiles along rays, u=1

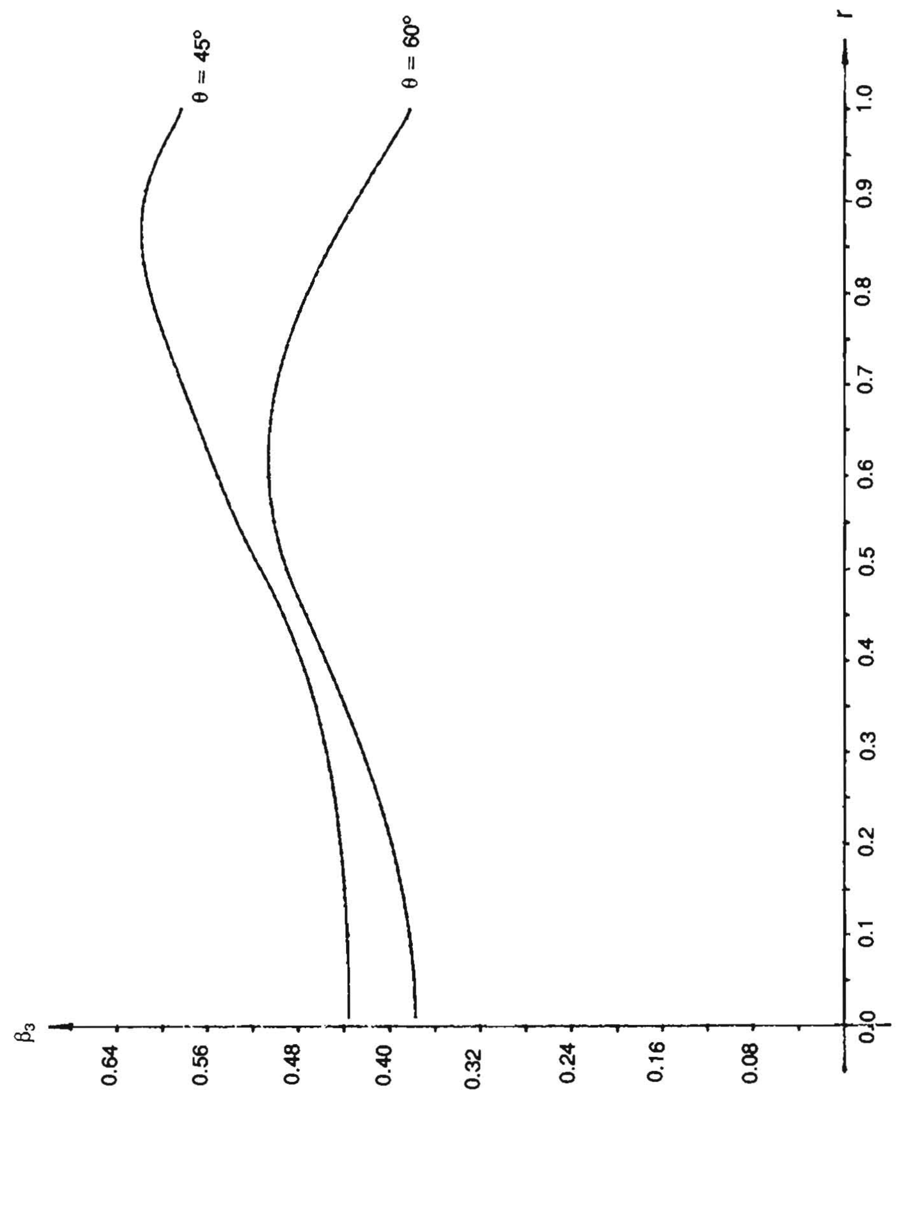

Fig. 11. Twisting moment profiles along rays, u=1

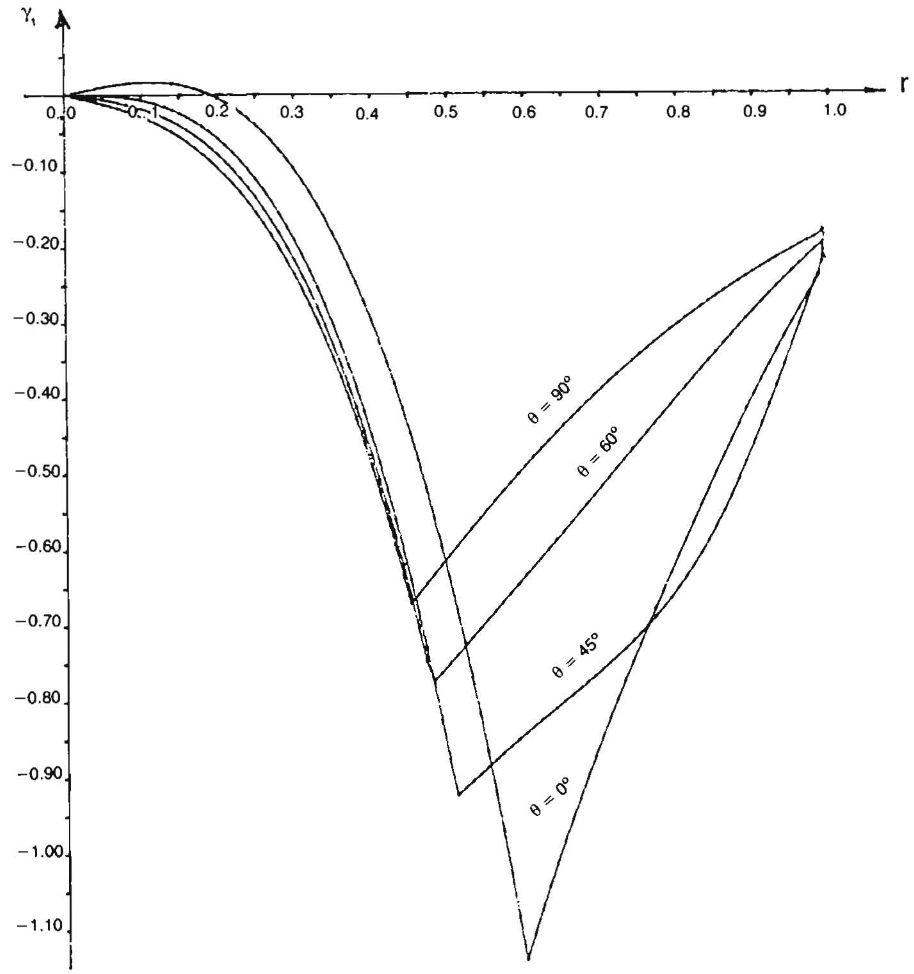

Fig. 12. Radial shear profiles along rays, u=1

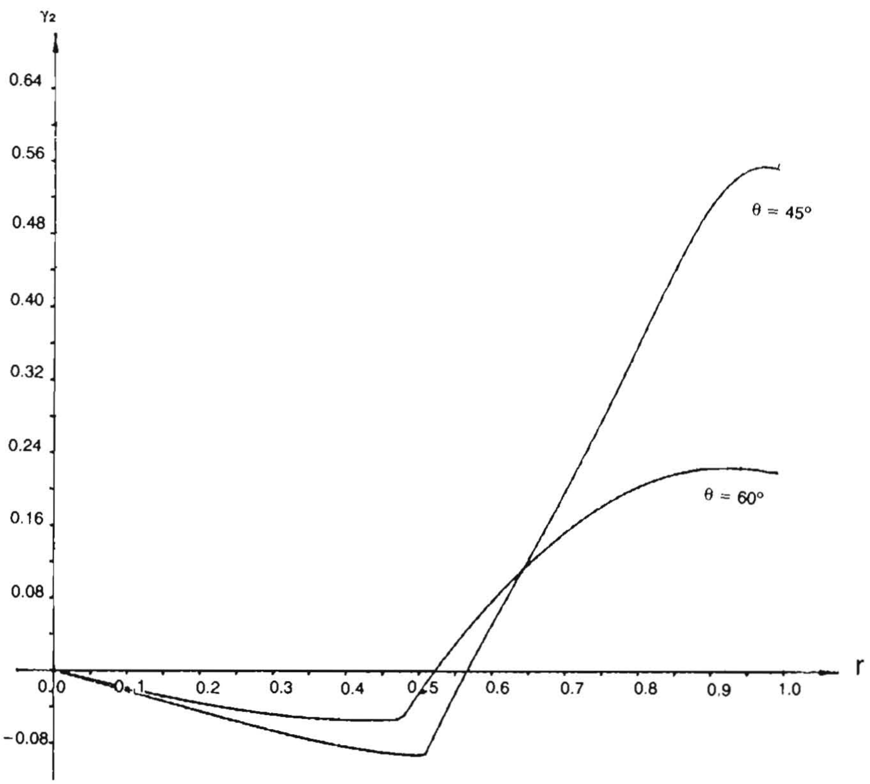

Fig. 13. Transverse shear profiles along rays, u=1

صفيحة دائرية رقيقة محملة بحمل تربيعي على قطع ناقص مركزي ومرتكزه ارتكازاً متماثلا عند ٤ نقاط

وديع عطاالله بسالي و محمد نعيم يحيى أنور

قسم الرياضيات ـ كلية العلوم ـ جامعة الكويت ـ الكويت

تم الحصول على صيغ مضبوطة لمتسلسلات لانهائيـة للإزاحـة العموديـة الصغيرة التي تحدث عند أية نقطة من صفيحة دائرية رقيقة محملة بحمل تربيعي موزع على مسـاحة قطع ناقص متحد مع الصفيحـة في المركـز، والصفيحة مـرتكزة إرتكـازاً متماثـلاً عند روءوس مستطيل أضلاعه موازية لمحوري القطع. لقد أعطيت أيضـاً صيغ مضبـوطة لعزوم الانثناء واللّي والقوى القاصة ودرست الحالات النهائية الآتية:

١ ـ حالة الصفيحة الكبيرة التي يؤول فيها نصف قطر الصفيحة الدائرية إلى ∞.

٢ ـ الحالة التي يؤول فيها القطع الناقص إلى دائرة مركزية.

٣ ـ الحالة التي يؤول فيها المحور الأصغر للقطع إلى الصفر.

لقد استخدم الحـاسب الإلكتروني في حـالتين خـاصتين للحصـول على جـداول عـددية لقيم الإزاحـة العموديـة وعزوم الانثنـاء واللّي والقـوى القاصـة، كـما رسمت منحنيـات لتمثيل تـوزيعها بيـانيـاً عـلى بعض أنصـاف الأقـطار في الـربـع الأول من الصفيحة.