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Integrability Conditions of A Structure Satisfying $f^5 - f = 0$

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ABSTRACT. In this paper, the integrability conditions of f(5,1)-structure satisfying $f^5 - f = 0$, where f is a non-null tensor field of type (1,1), are obtained. Besides this, a condition of complete integrability is presented.

1. Introduction

Let M be a C^∞, P-dimensional manifold, let $F \neq 0$ be a C^∞, tensor field of type (1,1), and of rank r, constant everywhere on M, such that

$$F^3 + F = 0 \quad , \quad 1 \leqslant r \leqslant n, \tag{1.1}$$

then M is called an F-structure manifold; (Yano 1963).

On an F-structure manidold, the operators

$$\text{(a) } \bar{\ell} = -F^2 \quad , \quad \text{(b) } \bar{m} = F^2 + 1 \tag{1.2}$$

applied to M_p, $P \in M$ are complementary projection operators, I is the identity operator. Since rank F = r, is constant everywhere, M, $\bar{\ell}$ and $\bar{m}$ give rise to two complementary differentiable distributions ℓ^* and m^* respectively.

The integrability conditions of ℓ^* and m^* were given by (Ishihara and Yano 1964).

On a differentialbe manifold M of dimension p, the structures defined by

$$(1.3) \qquad f^5 \pm f = 0$$

where $f \neq 0$ is a C^∞ tensor field of type (1,1) on M, were studied by (Andreou 1982).

For the structure satisfying

$$(1.4) \qquad f^5 - f = 0 \, ,$$

the operators given by

$$(1.5) \qquad (a)\ \ell = f^4 \quad , \quad (b)\ n = -f^4 + I$$

applied to M_p, $P \in M$ are complementary projection operators.

If rank f = r, is constant everywhere on M, then we have two complementary C^∞ distributions L and N on M corresponding to the operators ℓ and n respectively, dimension L = r, and dimension N = p-r. We also have

$$(1.6) \quad (a)\ \ell f = f\ell = f \quad , (b)\ fn = nf = 0, \Leftrightarrow \quad (c)\ f^2 n = nf^2 = 0$$

A structure defined by the characteristic equations (1.4) is called an f(5,1)-structure of rank r; (Bokhariyal 1985).

2. Integrability Conditions

A differentiable distribution D of a manifold M is said to be involutive if $[X,Y] \in D$, whenever the vector fields $X,Y \in D$. The distribution D is said to be integrable, if each pointof M lies in the domain of a flat chart. A distribution is integrable if and only if it is involutive (Brickell and Clark 1970).

Nijenhuis tensor of two C^∞-tensor fields A and B of type (1,1) is defined as

$$(2.1) \qquad N(X,Y) = \frac{1}{2}([AX,BY] + [BX,AY] + AB[X,Y] + BA[X,Y] - A[X,BY] - A[BX,Y] - B[X,AY] - B[AX,Y])$$

If $A = B$, we have

$$(2.2) \qquad N(X,Y) = [AX,AY] + A^2[X,Y] - A[AX,Y] - A[X,AY]$$

(Nijenhuis 1951).

We shall obtain the integrability conditions of the distributions L and N, using the Nijenhuis tensor for $A = B = f$.

Theorem 1

On an f(5,1)-structure manifold M we have

$$(2.3) \qquad (a)\ f^2\ell = \ell f^2 = f^2 \quad , \quad (b)\ f^4\ell = f^4 = \ell$$

i.e., f^2 acts on ℓ as an almost product structure, and on n as a null operator.

Proof

According to (1.6)a

(a) $f^2 = fo\ f\ell = f^2$. Similarly $\ell f^2 = f^2$
(b) $f^4\ell = f^3o\ f\ell = f^4 = \ell$. Similarly $\ell f^4 = \ell$,

i.e., f^2 acts on ℓ as an almost product structure. That f^2 acts on n as a null operator is clear, follows from (1.6)c.//

Theorem 2

A necessary and sufficient condition for the distribution L to be integrable is

$$(2.4) \qquad N(X,Y) = \ell n(X,Y)$$

Proof

L is integrable if and only if for any two vector fields X and Y we have

$$[\ell X, \ell Y] \in L \Leftrightarrow n\,[\ell X,\ell Y] = 0$$

Since $\ell + n = I$,

$$[\ell X,\ell Y] = \ell[\ell X,\ell Y] \qquad (1)$$

We have according to (2.3)

$$N(X,Y) = [fX,fY] + f^2[X,Y] - f[fX,Y] - f[X,fY]$$
$$\ell N(X,Y) = \ell[fX,fY] + f^2[X,Y] - f[fX,Y] - f[X,fY]$$

Since $\ell f = F$. Therefore

$$N(X,Y) - \ell N[X,Y] = [fX,fY] - \ell[fX,fY] \qquad (2)$$

If (2.4) is satisfied, we have

$$[fX,fY] = \ell[fX,fY] \qquad (3)$$

In (3), replace X,Y by f^3X,r^3,Y, we obtain (1), and L is integrable.

Conversely, assume that (1) is satisfied. Replacing X,Y by fX,fY, we obtain (3), which implies (2.4).//

Theorem 3

The following equations are equivalent

(2.5)
(i) $N(X,Y) = \ell N(X,Y)$
(ii) $nN(X,Y) = 0$
(iii) $nN(\ell X,\ell Y) = 0$
(iv) $nN(f^2X,f^2Y) = 0$

Proof

(i) $\ell + n = I$

$$N(X,Y) = \ell N(X,Y) \Leftrightarrow (I-\ell)N(X,Y) = 0 \Leftrightarrow nN(X,Y) = 0$$

Therefore, (i) $\Leftrightarrow$ (ii).

(ii) In $nN(X,Y) = 0$, replace X and Y by ℓX and ℓY respectively, we have

$$nN(\ell X,\ell Y) = 0$$

Therefore, (ii) $\Leftrightarrow$ (iii).

(iii) Since $nf^2 = 0$, $nf = 0$, and $f^5 - f = 0$,

$$
\begin{aligned}
& nN(\ell X,\ell Y) = 0 \Leftrightarrow nN(f^4X,f^4Y) = 0 \\
\Leftrightarrow\ & n([f^5X,f^5Y] + f^2[f^4X,f^4Y] - f[f^5X,f^4Y] - f[f^4X,f^5Y]) = 0 \\
\Leftrightarrow\ & n[fX,fY] = 0 \Leftrightarrow n[f^3X,f^3Y] = 0 \\
\Leftrightarrow\ & n([f^3X,f^3Y] + f^2[f^2X,f^2Y] - f[f^3X,f^2Y] - f[f^2X,f^3Y]) = 0 \\
\Leftrightarrow\ & nN(f^2X,f^2Y) = 0
\end{aligned}
$$

Therefore, (iii) $\Leftrightarrow$ (iv).

(iv) $nN(f^2X,f^2Y) = 0 \Leftrightarrow n(f^3X,r^3Y] = 0$

$\Leftrightarrow n[f^4X,f^4Y] = 0 \Leftrightarrow n[\ell X,\ell Y] = 0,$

by replacing X,Y with fX,fY. From Theorem 2, we have

$$m[[\ell X,\ell Y] = 0 \Leftrightarrow N(X,Y) = \ell N(X,Y).$$

Therefore, (iv) $\Leftrightarrow$ (i).//

Theorem 4

A necessary and sufficient condition for the distribution N to be integrable is

(2.6)
$$N(nX,nY) = 0$$

Proof

N is integrable if and only if for any two vector fields X and Y we have

$$
\begin{aligned}
&[nX,nY] \in N \Leftrightarrow \ell[nX,nY] = \Leftrightarrow f^2\ell[nX,nY] = 0 \\
N[nX,nY] &= [fnX,fnY] + f^2[nX,nY] - f[fnX,nY] - f[nX,fnX] \\
&= f^2[nX,nY] = f^2\ell[nX,nY]
\end{aligned}
$$

where $fn = 0$, and from theorem (1), $f^2\ell = f^2\ell$. Therefore N is integrable if and only if

$$N(nX, nY) = 0.//$$

We have $n + m = I$, then

(2.7)
$$
\begin{aligned}
N(X,Y) &= N(\ell X+nX, \ell Y+nY) \\
&= N(\ell X,\ell Y) + N(\ell X,nY) + N(nX,\ell Y) + N(nX,nY).
\end{aligned}
$$

Theorem 5

The distributions L and N are both integrable if and only if

$$N(X,Y) = \ell N(\ell X,\ell Y) + N(\ell X,nY) + N(nX,\ell Y) \tag{2.8}$$

Proof

Suppose that L and N are both integrable. It follows from Theorem 2 and Theorem 4

$$N(X,Y) = \ell N(X,Y), \quad N(nX,nY) = 0$$

Using equation (2.7) we obtain (2.8). Conversely, assume that (2.8) is satisfied. It follows from (2.7) that

$$N(\ell X,\ell Y) + N(nX,nY) = \ell N(\ell X,\ell Y).$$

Replacing X,Y by nX,nY, we obtain $N(nX,nY) = 0$,and so $nN(\ell X,\ell Y) = 0$. It follows from Theorem (3) that $N(X,Y) = \ell N(X,Y)$ and L and N are both integrable.//

3. Complete Integrability Condition

Suppose that the distribution L is integrable, denote by f_L the restriction of f to L, then $N(\ell X,\ell Y)$ is exactly the Nijenhuis tensor of f_L. It is not hard to see that f_L on each integral submanifold of L is integrable if and only if $N(\ell X,\ell Y) = 0$.

Similarly, suppose that the distribution N is integrable, denote by f_N the restriction of f to N, then $N(nX,nY)$ is exactly the Nijenhuis tensor of f_N, and from theorem (4) $N(nX,nY) = 0$.

Definition

A f(5,1)-structure manifold is said to be completely integrable if L and N are both integrable, and f_L on each integral submanifold of L is integrable.

Theorem 6

A necessary and sufficient condition that an f(5,1)-structure manifold be completely integrable is that

$$N(X,Y) = N(\ell X,nY) + N(nX,\ell Y). \tag{3.1}$$

Proof

An f(5,1)-structure manifold is completely integrable if and only if Nijenheuis tensors $N(\ell X,\ell Y)$ and $N(nx,nY)$ of f_L and f_N vanish simultaneously for any two vector fields X and Y on M. Since N(nX,nY) = 0, N is integrable; since N(ℓX,ℓY) = 0, we have nN((ℓX,ℓY) = 0 and L is integrable. It follows from (2.7) that (3.1) is satisfied.

Cnversely, assume that (3.1) is satisfied. It follows from (2.7) that

$$N(\ell X,\ell Y) + N(nX,nY)) = 0$$

Replacing X,Y by nX,nY, we have

$$N(nX,nY) = 0, \quad N(\ell X,\ell Y) = 0.//$$

4. Examples

1. Consider the 5-dimensional manifold M^5 on which

$$f = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad f^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $f^5 - f = 0$.

2. Consider the 6-dimensional manifold M^6 on which

$$f = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is easy to find out that $f^5 - f = 0$.

3. Consider M^5 on which

$$f = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 \\ 0 & 0 & 2x & 2y & 0 \end{bmatrix}, \quad f^2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 2x & 2y & 0 & 0 & 0 \end{bmatrix}$$

$$f^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2x & -2y & 0 & 0 & 0 \end{bmatrix}, \quad f^5 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2x & 2y & 0 \end{bmatrix} = f$$

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شروط تكاملية لبناء يحقق $f^5 - f = o$

عدنـــان العقيـــل

قسم الرياضيات ـ جامعة الكويت ـ الكويت

لقد أمكن، في هذا البحث، الحصول على شروط تكاملية لبناء f (5,1) يحقق $f^5 - f = o$ حيث إن f يمثل حقلا تنسوريا غير صفري من النمط (1,1). وكـذلك أمكن الحصـول على شرط لتكامل تام.