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On a Commutativity Question in Banach Algebras

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ABSTRACT. It is shown that the condition $\|xy\| \leq \alpha \|yx\|$ for all x and y, α being a positive constant, does not imply commutativity in nonunital complex Banach Algebras. Other results, concerning the structure of certain Banach algebras satisfying this condition are obtained.

1. Introduction

In this paper, we go further in the study - initiated in the second author's papers (Oudadess 1983) and (Oudadess 1984) - of complex Banach algebras satisfying the condition

$$\|xy\| \leq \alpha \|yx\| \qquad \text{(C)}$$

for all x,y in the algebra, where α is a positive constant. We show that condition (C) does not imply commutativity in nonunital complex Banach algebras - A counterexmple of this fact is given in Section 4.

If E is a complex Banach algebra satisfying (C), we know that $E^2 = \{xy | x,y \in E\}$ is contained in the centre of E (Oudadess 1983). As corollaries, the theorems of Le Page (1967) and Duncan and Tullo (1974) follow from this. Here, we extend the result of the latter to algebras with (not necessarily bounded) approximate identity and we show that if E is semi-simple or an integral domain, then E is commutative. We prove also that if E is not commutative, then any element of E is a divisor of zero.

Furthermore, we examine the case of simple and topologically simple complex Banach algebras satisfying (C). We obtain, in particular, a result of Gelfand - Mazur type.

Finally, we study finite dimensional Banach algebras satisfying (C). It is an algebra of this type which gives the counter-example of Section 4.

The formulation of theorem 2.3 is more general than the one in the first version.

2. General Case

We give some general properties of complex Banach algebras satisfying the condition (C).

Theorem 2.1.

Let E be a complex Banach algebra satisfying (C). Then

(i) $E^2 = \{xy \mid x,y \in E\}$ is contained in the centre of E.

(ii) If E admits a left (resp. right) approximate identity, then E is commutative.

Proof

(i) This result was obtained in (Oudadess 1983). It follows from Liouville's theorem since condition (C) implies that the holomorphic vector valued function

$$f(\lambda) = \exp(\lambda z).xy.\ \exp(-\lambda z)$$

where $\lambda \in \mathbb{C}$ and $x, y, z \in E$, is bounded.

(ii) Let $(e_i)_{i \in I}$ be an approximate identity of E. By (i), we have, for all $x, y \in E$

$$(xe_i)y = y(xe_i) \qquad (i \in I)$$

Whence $xy = yx$ □

Remarks

1. By (i) of the previous theorem, we have for any integer $n > 0$ and all $x, y \in E$

$$(xy)^n = x^n y^n$$

Thus, the spectral radius is submultiplicative on E and hence the set of quasi-nilpotent elements of E coincides with the radical of E (Oudadess 1983).

2. For any x, y $\in$ E, we have

$$(xy - yx)^2 = 0$$

Whence $xy-yx \in$ Rad E. Thus if E is semi-simple, E is commutative.

3. One could think that complex Banach algebras without approximate identity and satisfying (C) are either semi-simple or radical (Oudadess 1983). As the following example shows, this is not the case. Let B and D be complex Banach algebras, B commutative and non-radical. On the vector space A = B x D consider the product given by $(b_1, d_1)(b_2, d_2) = (b_1b_2, 0)$ and the norm $\| (b,d) \| = \| b \| + \| d \|$.

Since the projection on D of the product of any pair of elements of A is zero, A does not admit an approximate identity. Moreover $\rho_A((b,d)) . \rho_B(b)$, where ρ_A and ρ_B are spectral radius in A and B respectively; and $(0, d)^2 = 0$ for any d $\in$ D.

Thus A is a commutative complex Banach algebra (hence verifies (C)) without approximate identity which is neither semi-simple nor radical. □

Here is an interesting result on the structure of non commutative Banach algebras satisfying (C).

Theorem 2.2.

Let E be a complex Banach algebra satisfying (C). If E is not commutative, then any element of E is a two-sided divisor of zero.

Proof

Suppose that E is not commutative and note that condition (C) implies that any left (right) divisor of zero in E is two-sided. Let C (E) be the centre of E. We first show that if an element x of E is not a zero divisor, then x lies in C (E). Consider $z = xy - yx$ where y is any element of E. We have

$$xz = x(xy - yx) = 0 \quad \text{Whence } xy = yx \text{ for any } y \in E$$

i.e. $x \in C(E)$.

We now show that any element of C(E) is divisor of zero. Let $z \in C(E)$. Since E is not commutative, there exist x, y $\in$ E such that $xy - yx \neq 0$. Then $z(xy - yx) =$

z xy − z yx = zxy − xzy = zxy − zxy = 0. Thus any element of E is a divisor of zero. □

Remark

In a complex Banach algebra E satisfying (C), the centre of E cannot be a maximal ideal. Note that C (E) is an ideal by (i) of theorem 2.1. Let $x_o \in E - C(E)$. The subalgebra B of E generated by x_o and C(E) is a commutative subalgebra of E which is an ideal containing C(E) strictly. Hence C(E) is not maximal.

As a corollary of theorem 2.1 we have the following result which is an improvement of proposition II.3 in (Oudadess 1984).

Theorem 2.3.

Let E be a complex Banach algebra satisfying (C) and such that the subspace linearly spanned by $\{x.y : x,y \in E\}$ is dens in E. Then E is commutative.

3. Simple and Topologically Simple Algebras

We first deal with simple algebras.

Theorem 3.1.

Let E be a non trivial simple complex Banach algebra satisfying (C). Then E is isomorphic to $\mathbb{C}$.

Proof

$E_o = \{x \in E \mid xE = (0)\}$ is a two-sided ideal of E. Since E is simple and non trivial, $E_o = (0)$ *i.e.* for every $x \in E$, $x \neq 0$, we have $xE \neq (0)$. But for every $x \in E$, the set xE is a two-sided ideal of E. Whence xE = E for all non zero $x \in E$.This shows that E is commutative. To see that E is unital, let $x \in E$, $x \neq 0$.

There exists $1_x \in E$ such that $x1_x = x$. Then $E_x = \{y \in E \mid y1_x = y\}$ is a two sided ideal of E which is not reduced to (0). Hence $E_x = E$ and E is unital. Since xE = E for all non zero $x \in E$, the algebra E is a division algebra. The theorem follows from the classical Gelfand-Mazur Theorem. □

For topologically simple algebras, we have.

Theorem 3.2.

Let E be a non trivial topoligcally simple complex Banach algebra satisfying (C). Then E is a commutative integral domain.

Proof

The centre C (E) of E is a closed two-sided ideal of E such that $E^2 \subset C(E)$. Since E is not trivial and is topologically simple, we have C (E) = E, so E is commutative.

For any $x \in E$ consider $E_x = \{y \in E \mid xy = 0\}$. Since E_x is a closed ideal of E, we have either $E_x = E$ or $E_x = (0)$. If E was not an integral domain, the set J of elements x such that $E_x = E$ would be a closed two-sided ideal of E. But then J = E, *i.e.* E is trivial. This contradicts our assumption. □

4. Finite Dimensional Algebras

We now investigate the implications of condition (C) on the commutativity of finite dimensional algebras.

Theorem 4.1.

Let E be a finite dimensional complex Banach algebra satisfying (C). Then E is commutative if, and only if, Rad E is a commutative subalgebra of E.

Proof

By Wedderburn's theorem (Shilov 1977), we have $E = F \oplus \text{Rad } E$ (as vector spaces) where F is semi-simple subalgebra of E. Since F is semi-simple and finite dimensional, it admits a unit (Shilov 1977). Theorem 2.1. (ii) implies that F is commutative. Let $u \in F$ and $r \in \text{Rad } E$. We have x r = (xe)r = r(xe) = rx. Thus, any element of F commutes with all the elements of Rad E.

Consequently, E is commutative iff Rad E is a commutative subalgebra of E. □

In fact condition (C) does not imply commutativity in all complex Banach algebras as is shown by the following example:

Example 4.2.

Let e_1 and e_2 be two symbols verifying the relations $e_1^2 = e_2^2 = 0$ and $e_ie_je_i = 0$ (i, j = 1, 2) consider the complex algebra E generated by e_1 and e_2 subject to these relations. The algebra E is nothing else than the 4-dimensional vector space for which is a base is $\{e_1, e_2, e_1e_2, e_2e_1\}$.

For $x = x_1e_1 + x_2e_2 + x_3e_1e_2 + x_4e_2e_1$, put $\|x\| = \sum_1^4 |x_i|$. Then E is a non commutative radical Banach algebra satisfying $\|xy\| = \|yx\|$ for all $x, y \in E$.

Finally, we note that the counter example given in page 44 (Aupetit 1979) – in order to show that condition (C) does not imply commutativity in nonunital

Banach algebras - is not suitable since the subspace of M_3 (M_2 ($\mathbb{C}$)) considered there is not stable under product.

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حول مسألة الإبدالية في جبريات بناخ

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نتابع في هذا البحث الدراسة التي كان قد بدأها المؤلف الثاني حول جبريات بناخ على $\mathbb{C}$ التي تستوفي الشرط (C) التالي :

$$\| x y \| \leq \alpha \| y x \| \quad (C)$$

حيث α ثابتة موجبة و x و y عنصران أيا كانا في الجبر.

سنبرهن هنا على أن الشرط (C) لا يستلزم الإبدالية في حالة عدم إحتواء جبر بناخ على عنصر محايد حيث سنعطي مثالاً مضاداً في الفقرة الرابعة.

لتكن E جبر بناخ على $\mathbb{C}$. نعلم أنه إذا كانت E تستوفي الشرط (C) فإن $E^3 = \{ x.y \,;\, x,y \in E \}$ ضمن مركز E ، وبالتالي فإن مبرهنة لوباج ومبرهنة دانكان مع تولو تصبحان لازمتين لهذه النتيجة . وسنقوم في بحثنا هذا بتوسيع هاتين المبرهنتين لتشمل الجبريات التي تحتوي على عنصر محايد مقرب (غير محدود بالضرورة) . وسنبين أنه إذا كانت E شبه بسيطة أو كاملة فإنها تصبح بالضرورة إبدالية .

سنبين أيضا أنه إذا كانت E غير إبدالية ، فإن كل عنصر فيها قاسم لصفر .

سندرس كذلك حالة جبريات بناخ على D البسيطة أو البسيطة طبولوجيا والتي تستوفي الشرط (C). سنحصل خصوصا على مبرهنة من طراز كلفاند ـ ماز ور .

في نهاية هذا البحث، سندرس جبريات بناخ ذات البعد المنتهي والمستوفية للشرط (C) . وتجدر الإشارة إلى أن المثال المضاد الموجود في الفقرة الرابعة ـ من هذا البحث ـ هو من هذا الطراز.