Some Remarks on the Orientability of Spherical Fibrations*

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ABSTRACT. The purpose of this paper is to give several observations about the orientability of spherical fibrations with respect to generalized cohomology theories, and we shall give an example of a spherical fibration which is orientable with respect to mod p K-theory but not with respect to p-local K-theory.

The purpose of this paper is to give several observations about orientations of spherical fibrations with respect to generalized cohomology theories [3]. In general, we are concerned with the difference between E^*-orientability and $(E')^*$-orientability when given a map f: $E \rightarrow E'$ of commutative ring spectra. In particular, we shall explain on the classifying space level the fact that, away from 2, KO-orientability is equivalent to K-orientability and we shall give an example of a spherical fibration which is orientable with respect to mod p K-theory but not with respect to p-local K-theory, where p is an odd prime

We begin by recalling a standard result from the general theory of fibration sequences.

Lemma 1:

If the rows are fibre sequences and eg≃f in the diagram

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* Extracted from the Ph.D. Thesis.

$$\begin{array}{ccc}
F\widetilde{g} & \overset{\sigma}{\to} & FG \\
\downarrow & & \downarrow \\
\end{array}$$

$$\begin{array}{ccccccc}
\Omega X & \to & Ff & \to & Z & \overset{f}{\to} & X \\
\| & & \downarrow \widetilde{g} & & \downarrow g & & \| \\
\Omega X & \to & Fe & \to & Y & \overset{e}{\to} & X
\end{array}$$

and if $\widetilde{g}$ is constructed in the cononical way from a homotopy h: eg≃f (so that the middle sequare commutes and the left square homotopy commutes), then there exists a homotopy equivalence σ which makes the top square homotopy commute.

Now let E be a commutative ring spectrum and let B(SF;E) be the classifying space for E^*-oriented stable spherical fibrations, as constructed by May, Quinn, and Ray [5,III]. Let SFE be the component of $1\epsilon\pi_0 E$ in the zero[th] space of E. As explained by the cited authors, there is a fibration sequence of H-spaces and H-maps

(*) $SF \overset{e}{\to} SFE \overset{\tau}{\to} B(SF;E) \overset{q}{\to} BSF.$

Here BSF classifies stable spherical fibrations and q corresponds to neglect of orientation. This sequence is natural with respect to maps E, and the lemma has the following immediate consequence.

Proposition 2:

Let $f : E \to E'$ be a map of commutative ring spectra and let Z(f) denote the fibre of the induced map SFE → SFE'. Then there is a homotopy equivalence σ which makes the following diagram hopotopy commutative.

$$\begin{array}{ccc}
Z(f) & \overset{\sigma}{\to} & FB(SF;f) \\
\downarrow & & \downarrow \\
\end{array}$$

$$\begin{array}{ccccccc}
SF & \overset{e}{\to} & SFE & \overset{\tau}{\to} & B(SF;E) & \overset{q}{\to} & BSF \\
\| & & f \downarrow & & \downarrow B(SF;f) & & \| \\
SF & \overset{e}{\to} & SFE' & \overset{\tau}{\to} & B(SF;E') & \overset{q}{\to} & BSF
\end{array}$$

Thus the fibre Z(f) measures the difference between E^*-oriented and $(E')^*$-oriented stable spherical fibrations. It may happen that $f : SFE \to SFE'$ is equivalent to the inclusion of a direct factor.

Proposition 3:

Assume given a space Y(f) and a map g:Y(f) → SFE' such that the following composite is a week homotopy equivalence, where ϕ is the product

$$SFE \times Y(f) \overset{f \times g}{\longrightarrow} SFE' \times SFE' \overset{\phi}{\to} SFE'$$

Then the following composite is also a weak homotopy equivalence.

$$B(SF;E) \times Y(f) \xrightarrow{B(SF,f)\times\tau g} B(SF;E') \times B(SF,E';) \xrightarrow{\Phi} B(SF;E')$$

Proof:

For each i, the composite

$$\pi_i(Y(f)) \xrightarrow{g*} \pi_i(SFE') \xrightarrow{\partial} \pi_{i-1}(Z)(f))$$

must be an isomorphism. By the previous proposition, this isomorphism factors through $\pi_i(B(SF;E'))$ and the conclusion follows.

Of course, Y(f) should be thought of as a classifying space (or delooping) of Z(f). If E is an E_∞ ring spectrum, then (*) above is a sequence of infinite loop spaces and infinite loop maps and the delooping

$$Be : BSF \rightarrow BSFE$$

is the universal obstruction to E^*-orientability [5;V]. If E' is also an E_∞ ring spectrum and $f:E \rightarrow E'$ is an E_∞ ring map, then we have a commutative diagram

when the splitting of SFE' in Proposition 3 exists and deloops, the splitting of B(SF;E') also deloops. In this case, it is clear that a stable spherical fibration is $(E')^*$-orientable if and only if it is E^*-orientable. We can apply these considerations to localizations at a set of primes T (even though localizations of E_∞ ring spectra need not be E_∞ ring spectra) since, by [5,V.2.4], we have equivalences

$$B(SF;E_T) \simeq B(SF;E)_T \times BSF[T^{-1}]$$

and

$$B(SF;E_T)_T \simeq B(SF;E)_T.$$

We shall apply these observations to K-theory. As usual, let $B0_\otimes = SFKO$ and $BU_\otimes = SKF$. (Strictly speaking, we should be using the connective K-theory

spectra since only these are known to be E_∞ ring spectra). The real case is of particular interest since, as explained in [5,V§6], results of sulivan [6,V§6] imply an equivalance of fibration sequences

$$
\begin{array}{ccccccc}
SF & \to & F/PL & \to & BSPL & \to & BSF \\
\| & & \downarrow & & \downarrow & & \| \\
SF & \to & BO_\otimes & \to & B(SF;KO) & \to & BSF
\end{array}
$$

localized away from 2.

Henceforward, we agree once and for all to localize all spaces and spectra at a fixed odd prime p without change of notation.

Proposition 4.

There is an equivalence of infinite loop spaces

$$BU_\otimes \simeq BO_\otimes \times B(SU/SO) \times CP^\infty$$

Therefore there is an equivalence of infinite loop spaces

$$B(SF;K) \simeq B(SF;KO) \times B(SU/SO) \times CP^\infty$$

Proof:

At $P > 2$, $r \circ c : BO \to BO$ is an equivalnce and thus $BU \simeq BO \times B(U/O)$ as an infinite loop space. Of course, at $p > 2$ $BO \simeq BSO$ is 3-connected. On passege to 3-connected covers, we obtain an infinite loop equivalence

$$BSU \simeq BO \times B(SU/SO)$$

By Adams and Priddy [1], $BSU \simeq BSU_\otimes$ and $BSO \simeq BSO_\otimes$ as infinite loop spaces. By [5,V.3.1] $BU_\otimes \simeq BSU_\otimes \times CP^\infty$ as an infinite loop spaces.

This result correlates with the following observation.

Lemma 5:

Away from 2, an oriented S^k-fibration ξ (with section) over a CW-complex X is K^*-orientable if and only if it is KO^*-orientable.

Proof:

Complexification and realification give inverse isomorphisms $\widetilde{KO}^n(S^n) = \widetilde{K}^n(S^n)$. The conclusion follows from the definition of an orientation and the restriction to fibres diagrams

$$\widetilde{K}^n(T\xi) \longrightarrow \widetilde{K}^n(S^n)$$

$$r\downarrow \ \uparrow c \qquad\qquad r\downarrow \ \uparrow c$$

$$\widetilde{KO}^n(T\xi) \longrightarrow \widetilde{KO}^n(S^n)$$

Finally, we consider reduction mod p. Let $E/p = E\wedge MZ_p$ be the mod p reduction of E (where MZ_p is the mod p Moore spectrum) and let $f : E \to E/p$ be the natural map. Provided that E/p admits a structure of commutative ring spectrum such that f is a ring map, $Z(f)$ of proposition 2 measures the difference between E^*-oriented and $(E/P)^*$-oriented stable spherical fibrations. Since p is an odd prime, routine arguments from the results of Araki and Toda [2;II] and the absence of $\lim^1$ terms show that $K \to K/p$ and $KO \to KO/p$ are maps of commutative ring spectra. Of course K^*-orientability implies $(K/p)^*$-orientability. We show by an example that the converse fails.

Let $g : S^{2p-2} \to S^{2p-2}$ have degree p^2 and let
$$X = S^{2p-2} \ U_g \ e^{2p-2}$$

Then $\widetilde{H}^*(X;Z) = \widetilde{H}^{2p-1}(X;Z) \cong Z_{p^2}$ and therefore $\widetilde{H}^{2p-2}(X;Z_p) \simeq Z_p$ and

$$\widetilde{H}^{2p-1}(X;Z_p) \cong Z_p,$$

the generators being related by the second Bockstein β_2; of course, $\widetilde{H}^i(X;Z_p) = 0$ for all other i. We have the exact sequence

$$\cdots \to [S^{2p-1}, \ BSF] \to [X,BSF] \to [S^{2p-2}, \ BSF] \xrightarrow{p^2} [S^{2p-2}, \ BSF] \to \cdots$$

By Toda [7], $\pi_{2p-1}(BSF) = 0$ and $\pi_{2p-2}(BSF) = Z_p$. Thus there exists on S^k-fibration ξ over X of order p for k suitably large. Let $\bar{\xi}$ be the restriction of ξ to S^{2p-2}. The nontrivial integral cohomology groups of the Thom space $T\bar{\xi}$ are

$$\widetilde{H}^k(T\bar{\xi};Z) \cong Z \ \text{ and } \ \widetilde{H}^{k+2p-2}(T\bar{\xi};Z) \cong Z$$

We choose respective generators u and v whose mod p reductions $\bar{u}$ and $\bar{v}$ satisfy $P^1(\bar{u}) = \bar{v}$. Let q_1 and $\bar{q}_1$ be the first Wu classes of ξ and $\bar{\xi}$ and let ϕ be the Thom isomorphism in integral or mod p cohomology. Since $\bar{q}_1 = \phi^{-1} P^1 (\bar{u})$, $\bar{q}_1$ generates $H^{2p-2} (S^{2p-2};Z_p)$. Since q_1 restricts to $\bar{q}_1$, q_1 generates $H^{2p-2}(X;Z_p)$. If δ_p denotes the Bockstein homomorphism associated to

$$O \to Z \xrightarrow{P} Z \to Z_p \to 0$$

then the exact sequence

$$O \rightarrow H^{2p-2}(X;Z) \xrightarrow{p} H^{2p-2}(X;Z) \rightarrow H^{2p-2}(X;Z_p) \xrightarrow{\delta_p} H^{2p-1}(X;Z)$$

shows that $\delta_p(q_1)$ is p times a generator and is thus non-zero. It follows from [4, Prop 1.2] that the differential d_{2p-1} in the integral Atiyah - Hirzebruch spectral sequence for $T\xi$ is the composite

$$H^k(T\xi;Z) \rightarrow H^k(T\xi;Z_p) \xrightarrow{p^1} H^{k+2p-2}(T\xi;Z_p) \xrightarrow{\delta_p} H^{k+2p-1}(T\xi;Z).$$

Thus if $\triangle_\xi$ is the Thom class, then $d_{2p-1}(\triangle_\xi) = \phi(\delta_p(q_1)) \neq 0$.

However [4, Prop 1.2] implies that $d_{2p-1}(\triangle_\xi) = \phi(\beta q_1) = 0$ in the mod p Atiyah - Hirzebruch spectral sequence for $T\xi$, and there can be no further differentials for dimensional reasons. Therefore ξ is $(K/p)^*$-orientable but not K^*-orientable.

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بعض الملاحظات في توجه الألياف الكروية

تحسين غزال

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الغرض من هذا البحث إعطاء عدة ملاحظات حول توجه الألياف الكروية
بالنسبة لنظريات الكهومولوجي العامة. ان اهتمامنا بشكل عام يكمن في الفرق
بين التوجه في $^*E$ والتوجه في $^*(E')$ إذا أعطينا دالة $f: E \to E'^1$ بين أطياف حلقية
إبدالية. وبشكل خاص سنوضح على مستوى فضاءات التصنيف حقيقة أنه لغير
العدد ٢ فإن التوجه – KO مكافىء للتوجه في نظرية K . كذلك سنعطي مثالاً
لليف كروي قابل للتوجه بالنسبة لنظرية K قياس P ولكن غير قابل للتوجه
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