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Some Practical Results for Non-Exponential Models of Computer Performance Analysis

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ABSTRACT. Consistent measurements have shown that the service and interarrival time distributions for most computer resources are not exponential. The immediate consequence of this finding is that the popular Markovian models such as the M/M/1, M/M/C ···· etc. do not accurately represent the underlying computer resources and therefore may not be totally reliable in the performance analysis of computer systems. Computer resources may be more appropriately represented by non-exponential models where both the service and interarrival times may be of a general type. Although this seems to be a natural resort, in practice analysts refrain from the use of non-exponential models because they are hard to solve and may not lead to useful solutions.

This paper is an attempt to remove some of the difficulties associated with the analysis of non-exponential computer models. It is shown that the spectral solution of Lindley's integral equation with the help of Rouche's theorem may easily be used to obtain exact solutions for the non-exponential models of computer performance analysis. Several examples are used to illustrate the method. These include the $E_2/H_2/1$, the $H_2/H_2/1$ and the $E_2/E_2/1$ models for which exact performance measures are given for the first time, to the best of our knowledge.

1. Introduction

One of the basic models in the performance evaluation of computer systems is the single resource queueing model with infinite capacity and FCFS queueing discipline. Basic performance parameters of interest in this case are the mean waiting time, the expected number of jobs in the system and the probability distribution for the number of jobs in the system. In Markovian queueing theory, the determination of these parameters strongly depends on the characteristics of the interarrival and service time distributions. The analysis is easiest when both these distributions are of the exponential type, where in this case the performance metrics can be obtained using the well-known M/M/1 model (Kleinrock 1975).

However, in practice it has been shown that the exponential assumption is unrealistic and the M/M/1 may not truly resmelbe the actual system. In order to obtain better estimates for the performance metrics resort is made to more general models like the M/G/1, the G/M/1 or even the G/G/1 (for definitions, see reference (Kleinrock 1975). Unfortunately, the price paid for the generality in these models is a considerable mathematical complexity that is gravest in the case of the G/G/1 system. For this system, even the mean waiting time cannot be obtained analytically and approximations have to be used. To quote Kleinrock "We find ourselves in a difficult terrain when we enter the foothills of the G/G/1. Not even the average waiting time is known for this queue."

In this paper, it is shown that the G/G/1 problem can be largely overcome if the spectral solution of Lindley's integral equation (Smith 1953) is complemented by the power of Rouche's theorem (Levinson 1970). The spectral solution is originally due to Smith (Smith 1953) and gives the waiting time distribution in terms of a polynomial factorisation, that is very difficult to obtain.

In Section 2 the fundamental queueing theory results relating to Lindley's integral equation and its spectral solution are described. Analysis of the G/G/1 with rational Laplace transforms is considered in Section 3. Examples and applications are given in Section 4. Concluding remarks are given in Section 5. Throughout the study, the following notation will be adopted:

$A(t)$	the PDF of interarrival times
$a(t)$	the pdf of interarrival times
$A^*(\theta)$	the Laplace transform of a(t)
$\bar{a}$	the first moment of a(t)
$\bar{a}^2$	the second moment of a(t)
C_a^2	the coefficient of variation of interarrival times
C_s^2	the coefficient of variation of service times
E_k	Erlang_k distribution
$E(x)$	expected value of the random variable x
$F(t)$	the PDF of service times
$f(t)$	the pdf of service times
$F^*(\theta)$	the Laplace transform of f(t)
$\bar{f}$	the first moment of f(t)
$\bar{f}^2$	the second moment of f(t)
FCFS	first-come, first-served
H_2	$\text{hyperexponential}_2$ distribution
$I^*(\theta)$	the Laplace transform of the idle time distribution
$\bar{I}$	the first moment of the idle time distribution

$\overline{I}^2$	the second moment of the idle time distribution
<2>	the average number of jobs in the system
PDF	probability distribution function
pdf	probability density function
$W(t)$	the PDF of the waiting time distribution
$w(t)$	the pdf of the waiting time distribution
$W^*(\theta)$	the Laplace transform of $w(t)$
W	the mean waiting time
λ	mean arrival rate
μ	mean service rate
$\rho = \lambda/\mu$	the utilisation factor
σ_a^2	variance of interarrival times
σ_f^2	variance of service times

2. Basic Queueing Theory Results

Consider a stationary G/G/1 queueing system. Let t_{n+1} be the time between the nth and the (n+1)th arrivals in this queue, where t_{n+1} is drawn from the PDF $A(t)$ and $E(t_{n+1}) = 1/\lambda$. Let S_n be the service time of the nth customer, where S_n is drawn from the PDF $F(t)$ and $E(S_n) = 1/\mu$. Define $u_n = S_n - t_{n+1}$ and let $C(t)$ be the PDF of u_n. Then, if W_n is the waiting time (in queue) of the nth customer with PDF $W(t)$, we get

$$W_{n+1} = \max(0, W_n + u_n) \tag{1}$$

(Lindley 1952) has shown that the PDF for the random variable W_{n+1} is given by

$$W(t) = \int_{-\infty}^{t} W(t-u)\, d(C(u)) \quad t \geqslant 0 \tag{2}$$

which is valid for $\lambda/\mu < 1$.

A direct solution of (2) may not be feasible, but using spectrum factorisation (Kleinrock 1975) the following results can be obtained from (2). The Laplace transform of the waiting time pdf is given by:

$$W^*(\theta) = \theta\, \phi_+(\theta) \tag{3}$$

where

$$\phi_+(\theta) = K/\psi_+(\theta) \tag{4}$$

and $\psi_+(\theta)$ is obtainable from the factorisation

$$A^*(-\theta)F^*(\theta) - 1 = \psi_+(\theta)/\psi_-(\theta) \tag{5}$$

the factorisation is carried out such that $\psi_+(\theta)$, $\psi_-(\theta)$ satisfy the following conditions:

(i) $\psi_+(\theta)$ is an analytic function of θ with no zeros in the half-plane $Re(\theta) > 0$.

(ii) $\psi_+(\theta)$ is an analytic function of θ with no zeros in the half-plane $Re(\theta) > D$ where D is a constant $(D > 0)$.

Furthermore, if $A^*(-\theta)$, $F^*(\theta)$ are not to include any discontinuities and to be of finite moments, then (Kleinrock 1975) added the following two properties:

$$\text{(iii) For } Re(\theta) > 0, \quad \lim_{|\theta|\to\infty} \frac{\psi_+(\theta)}{\theta} = 1$$
$$\text{(iv) For } Re(\theta) > 0, \quad \lim_{|\theta|\to\infty} \frac{\psi_-(\theta)}{\theta} = -1 \tag{6}$$

The constant K is given by

$$K = \lim_{\theta\to 0} \psi_+(\theta)/\theta \tag{7}$$

It has also been shown (Marshall 1968) that K is actually the equilibrium probability that an arbitrary arrival finds an empty system. Denoting this probability as Π_0, we can write

$$\Pi_0 = K \tag{8}$$

(Marshall 1968) used the above results to generalise the Pollaczek-Khinchin (P−K) transform to give:

$$W^*(\theta) = \Pi_0(1-I^*(-\theta))/(1-A^*(-\theta)F^*(\theta)) \tag{9}$$

Furthermore, it has been shown by [Kleinrock 1975] that the average waiting time in the G/G/1 is given by

$$W = \frac{\sigma_a^2 + \sigma_f^2 + (\bar{a})^2 (1-\rho)^2}{2\bar{a}(1-\rho)} - \frac{\overline{I^2}}{2\bar{I}} \tag{10}$$

3. Analysis of the G/G/1 Queueing System with Rational Laplace Transforms

Although the results of section 2 have been available in the literature for some time now, analysts have so far refrained from applying them, mainly because of the difficulties arising in obtaining the factorisation (5). In this section it is illustrated how these difficulties can be overcome by the effective use of Rouche's theorem, which can be stated as:

Rouche's Theorem

"If $h(\theta)$ and $g(\theta)$ are analytic functions of θ inside and on a closed contour C, and if $|g(\theta)|<|h(\theta)|$ on C, then $h(\theta)$ and $h(\theta)+g(\theta)$ have the same number of zeroes inside C."

The system to be considered is a G/G/1 with arbitrary i.i.d. interarrival and service time distributions both assumed to have rational Laplace transforms. In this case, both $A^*(\theta)$ and $F^*(\theta)$ can be expressed as:

$$A^*(\theta) = \frac{NA^*(\theta)}{DA^*(\theta)} \qquad F^*(\theta) = \frac{NF^*(\theta)}{DF^*(\theta)}$$

where in each case the numerator and denominator are polynomials of the form $\sum_{n=0}^{L} b_n\theta^n$ where $\{b_n\}$ is a set of real coefficients and L is a positive integer.

3.1 The average number of jobs in the system

An explicit formula for the average number of jobs in the G/G/1 system described above can be obtained by employing the following theorems:

Theorem 1

For any G/G/1 model the ratio $\overline{I^2}/2\overline{I}$ can be expressed as

$$\frac{\overline{I^2}}{2\overline{I}} = -\frac{\psi_-'(0)}{\psi_-(0)} \tag{11}$$

where $\psi_-'(0) = d\psi_-(\theta)/d\theta|_{\theta=0}$ and

$\psi_-(\theta)$ is given by (5).

Proof

Using Marshall's generalisation of the P−K transform (9) it can be easily deduced that:

$$I^*(-\theta) = 1 + \theta/\psi_-(\theta) \tag{12}$$

Since $\overline{I}$ is the first moment of the idle time distribution, it clearly satisfies

$$\overline{I} = dI^*(-\theta)/d\,\theta|_{\theta=0}$$

Similarly, $\overline{I^2}$, the second moment, satisfies $\overline{I^2} = d^2I^*(-\theta)/d\theta^2|\ \theta = 0$.

Therefore, using (12) we get

$$\frac{\overline{I^2}}{2\bar{I}} = \frac{\psi_-'(0)}{\psi_-(0)} \qquad \text{Q.E.D.}$$

In view of this theorem, equation (10) becomes:

$$W = \frac{\sigma_a^2 + \sigma_f^2 + (\bar{a})^2(1-\rho)^2}{2\bar{a}(1-\rho)} + \frac{\psi_-'(0)}{\psi_-(0)}$$

Using the substitutions $\bar{a} = 1/\lambda$, $\sigma_a^2 = \frac{C_a^2}{\lambda^2}$, $\sigma_f^2 = \frac{C_s^2}{\mu^2}$, and knowing that $<n> = \lambda W + \rho$ (Little's formula) we get:

$$<n> = \frac{1+y_a+\rho^2 y_s}{(1-\rho)} + \lambda \frac{\psi_-'(0)}{\psi_-(0)} \tag{13}$$

where $y_a = (C_a^2-1)/2$, $y_s = (C_s^2-1)/2$. Moreover, since $F^*(\theta)$ is rational, then the LHS of the factorisation (5) can be rewritten as:

$$A^*(-\theta)F^*(\theta)-1 = \frac{NF^*(\theta)A^*(-\theta)-DF^*(\theta)}{DF^*(\theta)} \tag{14}$$

Theorem 2

For any G/G/1 with rational Laplace transforms, if the condition

$$|DF^*(\theta)| > |NF^*(\theta)A^*(-\theta)|$$

is satisfied on the closed semicircular contour that surrounds the left half of the θ-plane as shown in Fig. 1, then the average number of jobs in the system is given by:

$$<n> = \frac{1+y_a+\rho^2 y_s}{1-\rho} + \lambda \left\{ \frac{\frac{DA^{*\prime}(0)}{DA^*(0)} \prod_{k=i+2}^{m} \theta_k + B}{\prod_{k=i+2}^{m} \theta_k} \right\} \tag{15}$$

where

$$B = \begin{cases} 1 & m=i+2 \\ \sum_{k=i+2}^{m} \prod_{\ell=i+2,\ell\neq k}^{m} \theta_\ell & m>i+2 \end{cases}$$

m is the total number of zeros for the polynomial

$$NF^*(\theta)A^*(-\theta)-DF^*(\theta) = 0 \tag{16}$$

i is the total numer of zeros for the polynomial $\{DF^*(\theta)\}$ and $\theta_{i+2}, \theta_{i+3}, \cdots\cdots, \theta_m$ are all in $Re(\theta) > 0$.

Proof

By definition $\{DF^*(\theta)\}$ and $\{-NF^*(\theta)A^*(-\theta)\}$ are analytic on and inside the closed contour of Fig. (1). Therefore, if $|DF^*(\theta)|>| NF^*(\theta)A^*(-\theta)|$ then by Rouche's theorem $DF^*(\theta)$ and $\{NF^*(\theta)A^*(-\theta)-DF^*(\theta)\}$ will have the same number of zeros inside C. Since $\theta=0$ is always a zero, and $DF^*(\theta)$ has no zeros in $Re(\theta)>0$, $(F(t)=0, t<0)$, then the polynomial (16) will have only $m-i-1$ zeros in $Re(\theta)>0$, where i and m are as defined above. Denoting these zeros $\theta_{i+2}, \theta_{i+3}, \cdots, \theta_m$, the polynomial (16) can be factorised as follows:

$$\frac{\psi_+(\theta)}{\psi_-(\theta)} = \left\{\frac{NF^*(\theta)NA^*(-\theta) - DF^*(\theta)DA^*(-\theta)}{DF^*(\theta)\ (\theta-\theta_{i+2})\cdots(\theta-\theta_m)}\right\} \left\{\frac{(\theta-\theta_{i+2})\cdots(\theta-\theta_m)}{DA^*(-\theta)}\right\}$$

or, if all the roots are determined:

$$\frac{\psi_+(\theta)}{\psi_-(\theta)} = \left\{\frac{\theta(\theta_1+\theta)\cdots\cdot(\theta_i+\theta)}{DF^*(\theta)}\right\} \left\{\frac{(\theta-\theta_{i+2})\cdots(\theta-\theta_m)}{DA^*(-\theta)}\right\} \tag{17}$$

This gives

$$\psi_-(\theta) = \frac{DA^*(-\theta)}{(\theta-\theta_{i+2})\cdots(\theta-\theta_m)}$$

Differentiating and using (13) one gets (15). Q.E.D.

Theorem 3

For any stable G/G/1 with rational Laplace transforms, if the polynomial $\{NF^*(\theta)\}$ is of degree less than $\{DF^*(\theta)\}$ then the inequality:

$$|DF^*(\theta)|>| NF^*(\theta)A^*(-\theta)| \tag{18}$$

is satisfied in the closed, infinite radius semicircular contour C that encircles the left half of the $\theta-$plane as shown in Fig. 1.

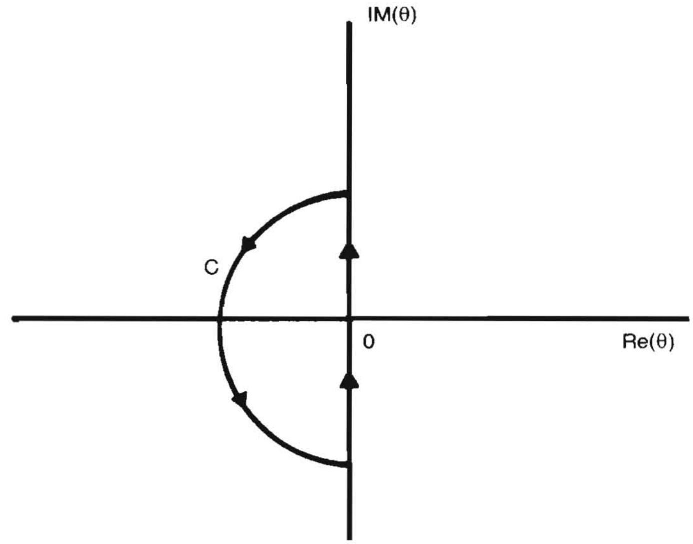

Fig. 1. The contour C

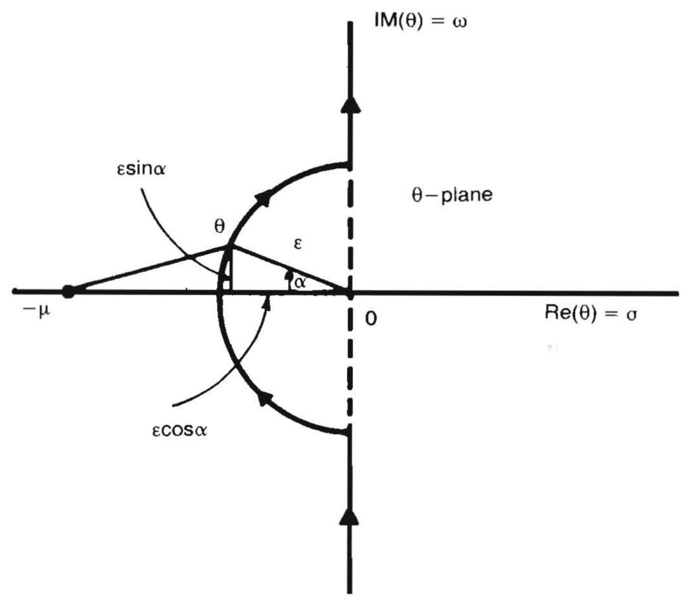

Fig. 2. The excursion around the origin

Proof

Let Re(θ) = σ and 1m(θ) = ω, in which case θ can be expressed as:

$$\theta = \sigma + j\omega\,, \qquad \text{where } j = \sqrt{1}$$

Clearly, for any point on C, σ<0, $e^{\sigma t} < 1$ (t > 0), and from complex analysis (Kleinrock 1975) $|e^{j\omega}| = 1$. Consequently, for any point θ on C,

$$\begin{aligned} |A^*(-\theta)| &= \left| \int_{0^-}^{\infty} e^{\theta t}\, dA(t) \right| \\ &= \left| \int_{0^-}^{\infty} e^{\sigma t}\, e^{j\omega t}\, dA(t) \right| \\ &\leqq \left| \int_{0^-}^{\infty} e^{j\omega t}\, dA(t) \right| \\ &\leqq \left| \int_{0^-}^{\infty} dA(t) \right| = 1 \end{aligned}$$

where $0^- = 0 - \varepsilon$

Therefore to prove the inequality:

$$|DF^*(\theta)| > |NF^*(\theta) A^*(-\theta)| \quad \text{on C}$$

it is enough to prove that

$$|DF^*(\theta)| > |NF^*(\theta)| \text{ on C}$$

This last inequality will hold if

$$\left| \frac{NF^*(\theta)}{DF^*(\theta)} \right| = |F^*(\theta)| < 1$$

i.e. to prove the inequality (18), we need only show that $|F^*(\theta)| < 1$ for any point on C.

First consider the case where σ < 0, which on C corresponds to the case where the radius R = |θ| is infinite (the semicircular curve). Taking the limit as $|\theta| \to \infty$ we get

$$\lim_{|\theta| \to \infty} | F^*(\theta) \quad = 0 < 1$$

which means that for |θ| very large $\left| \frac{NF^*(\theta)}{DF^*(\theta)} \right| < 1$, or $DF^*(\theta)| > | NF^*(\theta)|$ on the portion of C where σ < 0.

Secondly, consider the case where $\sigma = 0$, which corresponds to the vertical line overlapping with the imaginary axis. In this case:

$$|F^*(\theta)| < \int_{0-}^{\infty} |e^{-\theta t}| \quad |dF(t)|$$

$$= \int_{0-}^{\infty} |dF(t)|$$

$$= 1$$

But, by definition the Laplace transform $F^*(\theta)$ is equal to one only if $\theta=0$. Therefore,

$$|F^*(\theta)| < 1$$

and the inequality (18) holds for $\theta \neq 0$, $Re(\theta) = 0$.

At $\theta = 0$, apparently:

$$|DF^*(0)| = |NF^*(0)A^*(0)|$$

which contradicts (18). However, this problem can be overcome if the point $\theta = 0$ is replaced by the point $\theta = 0^-$, where

$0^- = 0-\varepsilon$, ε being a very small positive real number. At the point $\theta=0^-$, $\sigma=\varepsilon$, $\omega=0$, and

$$|F^*(0^-)| = \left| \int_{0-}^{\infty} e^{\varepsilon t} \, dF(t) \right|$$

$$= \left| \int_{0-}^{\infty} (1+\varepsilon t+ \frac{\varepsilon^2 t^2}{2I} + \cdots)dF(t) \right|$$

$$\cong \left| \int_{0-}^{\infty} (1+\varepsilon t+ 0(\varepsilon^2))dF(t) \right|$$

$$= \left| \int_{0-}^{\infty} dF(t) + \varepsilon t df(t)+0(\varepsilon^2) \right|$$

$$= (1+ \frac{\varepsilon}{\mu}) + 0(\varepsilon^2),$$

since the first moment of $f(t) = \frac{1}{\mu}$ *i.e.* at $\theta=0^-$

$$\left| \frac{NF^*(0^-)}{DF^*(0^-)} \right| \cong (1+ \frac{\varepsilon}{\mu})$$

or

$$|NF^*(0^-)| \cong |DF^*(0^-)| (1+ \frac{\varepsilon}{\mu})$$

Similarly, it can be shown that

$$|A^*(0^-)| = |\int_{0^-}^{\infty} e^{-\varepsilon t}\, dA(t)|$$

$$\cong (1 - \frac{\varepsilon}{\lambda} + 0(\varepsilon^2))$$

since the first moment of a(t) = $1/\lambda$

In view of this inequality (18) becomes:

$$|DF^*(0^-)| > |DF^*(0^-)|\ (1+\frac{\varepsilon}{\mu})\ (1-\frac{\varepsilon}{\lambda})$$

$$= |DF^*(0^-)|\ (1- \frac{\varepsilon(\mu-\lambda)}{\lambda} + 0\ (\varepsilon^2))$$

$$= |DF^*(0^-)|\ (1- \frac{\varepsilon(1-\rho)}{\lambda} + 0\ (\varepsilon^2))$$

which is satisfied for any stable G/G/1 ($\rho < 1$).

This inequality is even stronger if $1m(\theta) = \omega = \varepsilon$. Therefore inequality (18) is satisfied on all points of the contour C.

Remarks

1. The motivation behind considering the left half of the θ−plane is the knowledge that $F^*(\theta)$ has no poles in $Re(\theta) > 0$, and consequently all the zeros of $\{DF^*(\theta)\}$ are in $Re(\theta) < 0$.

2. A development similar to theorem 3 has been conducted in (Kleinrock 1975) for the particular case of the G/M/1 system.

For examples and applications, see Section 4.

3.2 The Waiting Time Distribution

For the G/G/1 system described in theorem 3, the waiting time distribution can now easily be obtained using (3), (4), (7) and (17).

By (3) and (4)

$$W^*(\theta) = \frac{K\theta}{\psi_+(\theta)}$$

where by (7) and (17)

$$K = \frac{\prod_{\ell=1}^{i} \theta_\ell}{DF^*(0)}$$

Therefore, using (17):

$$W^*(\theta) = \frac{DF^*(\theta) \prod_{\ell=1}^{i} \theta_\ell}{DF^*(\theta) \prod_{\ell=1}^{i} (\theta_\ell + \theta)} \tag{19}$$

For any particular system this formula can be inverted using partial fraction expansion to give the waiting time distribution.

The mean waiting time can now be obtained using the relation:

$$W = - \left. \frac{dW^*(\theta)}{d\theta} \right|_{\theta=0}$$

which gives

$$W = \frac{\frac{DF'^*(0)}{DF^*(0)} \prod_{\ell=1}^{i} \theta_\ell - D}{\prod_{\ell=1}^{i} \theta_\ell} \tag{20}$$

where

$$D = \begin{cases} 1 & \text{if } i=2 \\ \sum_{k=1}^{i} \prod_{\ell=1, \ell \neq k}^{i} \theta\ell & \text{if } i > 1 \end{cases} \tag{21}$$

and $\theta, \cdots, \theta_i$ are all in $Re(\theta) > 0$.

3.3 Another Formula for $<n>$

Using Little's formula, $<n> = \lambda T = \lambda T = \lambda W + \rho$, one can now easily obtain another expression for the average number of jobs in the system as:

$$<n> = \frac{\lambda \{(DF'^*(0) \prod_{\ell=1}^{i} \theta_\ell - DF^*(0)D\} + \rho DF^*(0) \prod_{\ell=1}^{i} \theta_\ell}{DF^*(0) \prod_{\ell=1}^{i} \theta_\ell} \tag{22}$$

where D is given by (21).

This formula is equivalent to (15) and can be used if the number of zeros in $Re(\theta) < 0$ is less than that in $Re(\theta) > 0$. (Evaluating less zeros saves effort).

Remarks

For the special case of the M/G/1 it is known that the idle time distribution is exponential and $\overline{I^2}/2\overline{I} = 1/\lambda$. Therefore (13) reduces to the famous P−K mean formula:

$$<n> = \frac{\rho(1+\rho y_s)}{1-\rho}$$

which applies for any G-type service type distribution, rational or not.

In general, theorem 3 applies for any G-type inter-arrival time distribution and there is no need to impose the rationality condition on $A^*(\theta)$. However to obtain a numerical value for $<n>$ one needs an explicit form for the polynomial (16).

4. Applications and Examples

To illustrate the methodology developed above, some examples are considered in this section:

1) The Average Number of Jobs in the $E_2/H_2/1$

For the $E_2/H_2/1$, the interarrival time distribution is an Erlang-2 and the service time distribution is a hyperexponential-2. Therefore,

$$A^*(\theta) = \left(\frac{2\lambda}{2\lambda+\theta}\right)^2 \tag{23}$$

and

$$F^*(\theta) = \frac{\alpha_1 \mu_1}{\mu_1+\theta} + \frac{\alpha_2 \mu_2}{\mu_2+\theta}$$

$$= \frac{\mu_1\mu_2 + (\alpha_1\mu_1 + \alpha_2\mu_2)\theta}{\mu_1\mu_2 + (\mu_1+\mu_2)\theta + \theta^2} \tag{24}$$

From this

$$NF^*(\theta) = \mu_1\mu_2 + (\alpha_1\mu_1 + \alpha_2\mu_2)\theta$$

$$= \mu_1\mu_2 + (\mu_1+\mu_2)\theta - \frac{\mu_1\mu_2}{\mu}\theta \tag{25}$$

and

$$DF^*(\theta) = \mu_1\mu_2 + (\mu_1+\mu_2)\theta + \theta^2 \quad (26)$$

which is of degree higher than $NF^*(\theta)$, implying that theorem 3 applies. In view of this, the polynomial (16) becomes:

$$(\mu_1\mu_2+(\mu_1+\mu_2)\theta+\theta^2) - (\mu_1\mu_2+(\mu_1+\mu_2)\theta - \frac{\mu_1\mu_2}{\mu}\theta)A^*(-\theta) = 0$$

or, using (23)

$$(\mu_1\mu_2+(\mu_1+\mu_2)\theta+\theta^2) - (\mu_1\mu_2+(\mu_1+\mu_2)\theta - \frac{\mu_1\mu_2}{\mu}\theta)(\frac{\lambda}{(2-\theta)} = 0 \quad (27)$$

Clearly, this polynomial has a total of 4 zeros, one of which is $\theta=0$. Since $\{DF^*(\theta)\}$ has two zeros in $Re(\theta) < 0$, it is implied that two of the remaining 3 zeros for (27) lie in $Re(\theta) < 0$, and only one zero, θ_4, say, is in $Re(\theta) > 0$.

Applying equation (15), we get:

$$<n> = \left\{\frac{1+y_a + \rho\,^2y_s}{1-\rho}\right\} + \lambda\frac{\{\frac{DA'^*(0)}{DA^*(0)}\theta_4+1\}}{\theta_4}$$

But from (23)

$$\frac{DA'^*(0)}{DA^*(0)} = -\frac{1}{\lambda}$$

Therefore

$$<n> = \left\{\frac{1+y_a + \rho\,^2y_s}{1-\rho}\right\} + \frac{\lambda-\theta_4}{\theta_4} \quad (28)$$

Equation (28) gives the exact value for the average number of jobs in the $E_2/H_2/1$ if θ_4 is determined. θ_4 can easily be determined by applying the Newton-Raphson method to the polynomial:

$$\theta^3-\theta^2\{4\lambda-\mu_1-\mu_2\} + \theta\,\{4\lambda^2-4\lambda(\mu_1+\mu_2)+\mu_1\mu_2\} - 4\lambda\mu_1\mu_2(1-\rho)=0$$

which is obtained by manipulating (27).

2) The $E_2/E_2/1$

In this case:

$$A^*(\theta) = (\frac{4\lambda}{2\lambda+\theta})^2, \quad F^*(\theta) = (\frac{2\mu}{2\mu+\theta})^2$$

Therefore $NF^*(\theta) = 4\mu^2$, $DF^*(\theta) = 4\mu^2+4\mu\theta+\theta^2$ and consequently polynomial (16) becomes:

$$(4\mu^2 + 4\mu\theta + \theta^2) - 4\mu^2 A^*(-\theta) = 0 \tag{29}$$

Since $NF^*(\theta)$ is of degree less than $DF^*(\theta)$, theorem 3 applies.

implying that (24) has only two zeros in $Re(\theta) < 0$. Since $\theta=0$ is always a zero and the total number of zeros = 4, polynomial (29) will have only one zero θ_4, say, in $Re(\theta) > 0$. It can also be shown that

$$\frac{DA'^*(0)}{DA^*(0)} = -\frac{1}{\lambda}$$

and consequently (15) gives $<n>$ againt as:

$$<n> = \frac{1+y_a + \rho^2 y_s}{1-\rho} + \frac{\lambda-\theta_4}{\theta_4}$$

where θ_4 can be obtained by applying the Newton-Raphson method to the polynomial:

$$\theta^3+4\theta^2(\mu-\theta) + 4\theta(\mu^2-4\mu\lambda+\lambda^2) - 16\lambda\mu(\mu-\lambda) = 0$$

which is obtained by manipulating (29).

3) The $H_2/H_2/1$

In this case

$$A^*(\theta) = \frac{\gamma_1\lambda_1}{\lambda_1+\theta} + \frac{\gamma_2\lambda_2}{\lambda_2+\theta}, \quad F^*(\theta) = \frac{\alpha_1\mu_1}{\mu_1+\theta} + \frac{\alpha_2\mu_2}{\mu_2+\theta}$$

for which

$$NF^*(\theta) = \mu_1\mu_2 + (\mu_1+\mu_2)\theta - \frac{\mu_1\mu_2}{\mu_2}\theta$$

$$DF^*(\theta) = \mu_1\mu_2 + (\mu_1+\mu_2)\theta + \theta^2$$

and polynomial (16) becomes:

$$\{\mu_1\mu_2 + (\mu_1+\mu_2)\theta+\theta^2\} - \{\mu_1\mu_2 + (\mu_1+\mu_2)\theta - \frac{\mu_1\mu_2}{\mu}\} A^*(-\theta) = 0 \tag{30}$$

Since $\{NF^*(\theta)\}$ is of degree less than $\{DF^*(\theta)\}$, theorem 3 applies.

implying that (25) has only two zeros in $Re(\theta) < 0$. Again this leaves us with only one zero, θ_4 in $Re(\theta) > 0$, since $\theta=0$ is always a zero and the total number of zeros $= 4$. It can also be shown that

$$\frac{DA'^*(0)}{DA^*(0)} = \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}$$

and therefore (15) reduces to

$$<n> = \frac{(1+y_a + \rho\, ^2y_s)}{1 - \rho} + \frac{\lambda}{\theta_4} + \frac{\lambda(\lambda_1+\lambda_2)}{\lambda_1\lambda_2} \tag{31}$$

where θ_4 is determined as the only zero in $Re(\theta) > 0$, for the polynomial:

$$\theta^3 + \theta^2\{\mu_1+\mu_2-\lambda_1-\lambda_2\} + \theta\ \{\lambda_1\lambda_2+\mu_1\mu_2-(\lambda_1+\lambda_2)(\mu_1+\mu_2) +$$

$$(\alpha_1\mu_1+\alpha_2\mu_2)\ (\gamma_1\lambda_1+\gamma_2\lambda_2)\} - \frac{\mu_1\mu_2\lambda_1\lambda_2}{\lambda}(1-\rho) = 0$$

which is obtained by manipulating (30).

A Numerical Example

In Table 1, the average number of jobs in an $E_2/H_2/1$ system have been computed using equation (28). The results are compared with some simulation results given in (Gelenbe 1980). It can easily be observed that the simulation results can be highly credible. Similar computations can be carried out for other systems like the $E_2/E_2/1$ and the $H_2/H_2/1$.

Table 1. The average number of jobs in the $E_2/H_2/1$ system compared with simulation

C_a^2	ρ	C_s^2	$E_2/H_2/1$ <n> SIM	$E_2/H_2/1$ <n> exact
0.5	0.75	2	3.44 ± 0.05	3.440
		4	5.67 ± 0.12	5.684
		8	10.08 ± 0.32	10.18
		16	19.27 ± 0.83	19.177
		32	37.39 ± 1.92	37.176
		64	73.02 ± 4.73	73.175
		128	146.00 ± 14	145.175
	0.8	2	4.67 ± 0.09	4.667
		4	7.83 ± 0.22	7.861
		8	14.11 ± 0.43	14.257
		16	27.24 ± 1.39	27.054
		32	52.95 ± 3.02	52.653
		64	102.4 ± 8.0	103.853
		128	203.7 ± 21.0	206.252

5. Conclusion

Spectral methods and Rouche's theorem have been combined to obtain exact expressions for the waiting time distribution and the average number of jobs in any G/G/1 system with rational Laplace transforms. The methodology has been illustrated by considering the $E_2/H_2/1$, the $E_2/E_2/1$ and the $H_2/H_2/1$ systems.

The analysis solves a longstanding practical problem, reduces the need for approximations and expensive simulations. Moreover, it allows the analyst to evaluate the effect of service and interarrival time distributions on the performance metrics.

The present methodology is short of providing an expression for the probability distribution of the number of jobs in the G/G/1 system. However, this problem will be considered separately in a sequel paper.

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بعض النتائج العملية للنماذج الصفية العامة المستخدمة في أداء الحاسبات الآلية

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تستخدم النماذج الصفية بكثرة الآن في تحليل أداء نظم الحاسبات الآلية وتصميمها. إن العناصر ذات الأهمية في هذا المجال هي متوسط عدد البرامج الذي ينتظر عند جهاز معين والزمن الذي ينتظره برنامج معين لأداء مهمة على أحد الأجهزة وكمية البرامج التي يكملها الحاسب في فترة زمنية محددة.

وقد جرت العادة أن يستخدم المحللون نماذج صفية ماركوفية في تحليل أداء الحاسبات مثل نماذج م/م/١، م/م/ج... الخ وذلك تجنباً للتعقيدات الرياضية الناتجة في غيرها من النماذج. غير أن التجارب دلت مؤخراً على أن هذه النماذج لا تمثل الحاسبات الآلية بالدقة الكافية إذ أنها تفترض أن التوزيع الإحتمالي للبرامج الداخلية والخارجة توزيع تصاعدي ـ وهذا الإفتراض لا يسنده الواقع. ونتيجة لهذا لزم الإتجاه إلى نماذج عامة أكثر تمثيلاً للحاسبات الآلية مثل نموذج ع/ع/١، أ/هـ/١... الخ. بيد أن أستخدام هذه البرامج ليس بالأمر الميسور، فالكثير منها محاط بسياج من التعقيدات الرياضية ـ بل أن بعضها يستحيل حله رياضياً.

في هذه الورقة نحاول تذليل التعقيدات الرياضية المتعلقة بالنماذج الصفية العامة بحيث تصبح صالحة للإستخدام في تقييم أداء الحاسبات الآلية. يعتمد العمل في الورقة على أربع دعامات:

* معادلة لندلي التكاملية.
* الحل الطيفي لمعادلة لندلي.
* نظرية روش.
* النتائج التي حصل عليها كلاينروك.

البحث في هذه الورقة يثبت نظريتين أساسيتين يترتب عليهما إمكانية الحصول على الحل الطيفي لمعادلة لندلي التكاملية المتعلقة بقطاع واسع من نماذج ع/ع/١ وبالتالي الحصول على معادلات حقيقية للقيم المطلوبة في تحليل أداء الحاسبات الآلية مثل متوسط زمن الإنتظار ومتوسط البرامج المنتظرة عند جهاز ما .

يختتم البحث بتطبيق الطرق والمعادلات لإعطاء نتائج لم يحصل عليها من قبل (حسب علمنا)، وذلك مثل النتائج المتعلقة بنظام أ/هـ/١ ، أ/أ/١ ، هـ/أ/١ وأخيراً نوضح دقة النتائج بمثال عددي .