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A Comparative Study of the Frequency Responses of Some Existing Three-Dimensional Second Derivative Coefficient Sets with New Improvements

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ABSTRACT. The frequency responses of the all three-dimensional second derivative coefficient sets derived by Rao *et al.* (1970), along with that of their simple formula and the theoretical response of the second derivative operation, using an infinite number of points average approach, are presented. The frequency responses of the coefficient sets reveal: (a) the superiority of the coefficient sets derived following Peters' approach over those sets derived following Elkins' approach, (b) the superiority of the coefficient sets derived with a weightage of l/r^4 to all circles over those derived with a weightage of l/r^2 or without any weightage to all circles, (c) that many coeficient sets derived following Peters' approach give more accurate results than the simple formula, and also (d) the coefficient sets derived with preference to central point give better results than those derived with no preference to the central point.

However, for the calculation of the second derivative, two new weight coefficient sets which use the least possible number of circles for obtaining average values and at the same time yield good results are developed by making use of Richardson's improvement formula of the derivative. We also present a comparative picture of the frequency responses of the derived sets, along with that of the best set derived by Rao *et al.*

The second derivative method of interpreting gravity data, although its use is justifable only data of high accuracy, offers a simple routine method of locating some types of geologic anomalies of importance in oil and mineral exploration. The analytical method of calculation of the second vertical derivative involves summation of a number of products of the average gravity values over circles of different radii with their corresponding weight coefficients. Weight coefficient sets have been proposed by Peters (1949), Henderson and Zietz (1949), Elkins (1951), Baranov (1953), Rosenbach (1953), and Rao *et al.* (1970). In all the methods, the average gravity value $\bar{g}(r)$ over a circle of radius r is expressed in terms of an infinite series as

$$\bar{g}(r) = a_o + a_2r^2 + a_4r^4 + a_6r^6 + \cdots \tag{1}$$

where

$$a_o = \bar{g}(O) \ , \ a_2 = -1/4 \frac{\partial^2 g}{\partial z^2} \ , \ a_4 = 1/64 \frac{\partial^4 g}{\partial z^4} \ , \ \cdots$$

For calculation of the second derivative, the infinite series is truncated after a certain number of terms and the resulting expression is solved either by a least-squares technique or by evaluation of simultaneous equations in terms of the averages of gravity values over circles of different radii.

The grid systems used in the investigation carried out by Rao *et al.* (1970) consist of circles as defined by Peters (1949). They differ from one another in the number of circles and their radii. The number of ring systems used in their investigation is 5. Six sets of coefficients are obtained for a given ring system under Peters' approach. Similar sets of coefficients derived by Elkins' approach are also obtained by Rao *et al.* for a given ring system. Thus in total 60 sets of coefficients are obtained. They also derived a simple formula which utilizes the average of the anomalies around a single circle and the anomaly at the central point. In all cases, except in the case of the simple formula, the coefficients are derived by a least-squares technique. The numerical values of the derived sets of coefficients are given in Table 1 and 2 in Rao *et al.* (1970).

The relative accuracy of the coefficient sets thus obtained was examined by Rao *et al.* (1970) as follows:

The gravity anomalies of two spheres of depth 4 units and 8 units, respectively, are calculated. The coefficient sets are then applied to the anomalies thus calculated at four different points, one immediately above the center of the body and the other at distances of 4,8, and 12 units from the origin. The percentage errors are then obtained from the theoretical and the calculated second derivative values at the above mentioned points. It is evident that Rao *et al.* technique is highly subjective because the results depend only on a few points of the plane of observation and on specific types of bodies with limiting depths.

The second derivative operation on gravity or magnetic data acts as a numerical filter. Therefore, frequency analysis makes it possible to judge the accuracy of a coefficient set by matching its amplitude response with the theoretical amplitude response.

The purpose of this paper is to judge the accuracy of all three-dimensional second derivative coefficient sets derived by Rao *et al.* (1970) (in total 60 sets of coefficients) using the method of frequency analysis; and to develop, for the calculation of the second derivative, weight coefficient sets which use the least possible number of circles for obtaining average gravity values and at the same time yield good results. We have found suitable weight sets by making use of Richardson's formula (Maron 1982). We also present a compartive picture of the frequency responses of the derived sets, along with those of the existing coefficient sets.

Richardson's Improvement Formula

Richardson's formula can be used to produce an improved estimate from two known estimates. This important formula is described in Maron (1982); it is applied to the approximations of derivatives and integrals as well as to the solution of differential equations. Here, we show how to use it to obtain an improved approximation of the second derivative of the gravity field from two known approximations of the field.

Suppose that $g_{zz}(h)$ is an $O(h^n)$ approximation of the exact second derivative of gravity g_{zz} and we use it to get two approximations $g_{zz}(h)$ and $g_{zz}(h_{larger})$, where h is the grid spacing. Then an improved approximation of the exact second derivative (g_{zz}) is given by Richardson's formula as

$$\big(g_{zz}(h)\big)_1 = \frac{q^n\, g_{zz}(h) - g_{zz}(h_{larger})}{q^n - 1} \tag{2}$$

where

$$q = \frac{h_{larger}}{h},$$

Since $g_{zz}(h)$ is an $O(h^n)$ approximation of g_{zz}, then $g_{zz}(h)_1$ is mth order, that is,

$$g_{zz} - \big(g_{zz}(h)\big)_1 = O(h^m) \tag{3}$$

In this case we can use (2) to get still higher-order approximations (Maron 1982),

$$\big(g_{zz}(h)\big)_2 = \frac{q^m\big(g_{zz}(h)\big)_1 - \big(g_{zz}(h_{larger})\big)_1}{q^m - 1} \tag{4}$$

Thus, knowing $g_{zz}(h)$, q, n, and m, one can obtain improved approximations of the exact second derivative g_{zz} by making use of Richardson's improvement formula. Note that Richardson's formula can be used whenever we know n, even if we do not know m (Maron 1982). In this important formula, n depends only on the approximating formula $g_{zz}(h)$, where q depends on the two stepsizes used.

Notation

Following Rao *et al.* (1970), the following notation has been used throughout:

Elkins' type: The coefficient sets derived by Rao *et al.* representing the average radial gravity $\bar{g}(r)$, as

$$\bar{g}(r) = a_o + a_2 r^2 \tag{5}$$

Peters' type: The coefficient sets representing the average radial gravity:

$$\bar{g}(r) = a_o + a_2 r^2 + a_4 r^4 \tag{6}$$

P: Preference to central point: The coefficients derived by replacing a_o by $\bar{g}(O)$ in equations (5) and (6).

N: No preference to central point.

$1/r^2$:The coefficients derived where the average radial gravity for each circle is given a weightage of $1/r^2$ except in the case of $\bar{g}(O)$, the anomaly at the origin, which is given a weightage of unity.

$1/r^4$: The coefficients derived where the average radial gravity for each circle is given a weightage of $1/r^4$ except ... etc.

1 : The coefficients derived without any weightage to any circle.

S or S $O(h^2)$: Coefficients derived utilizing the average of the anomalies around a single circle and the anomaly at the central point.

S $O(h^4)$: The first improved coefficients derived using Richardson's formula.

S $O(h^6)$: The second improved coefficients derived using Richardson's formula.

$O(h^n)$: The power of terms neglected in the approximation.

h : The spacing between stations.

g_{zz}: Exact second vertical derivative of gravity.

$g_{zz}(h)$: Approximate second vertical derivative of gravity.

u/2 and v/2: The frequencies in cycles per unit of length in the x and y directions respectively, where $\rho^2 = u^2 + v^2$.

Frequency Analysis of Previously Proposed Set of Weights with new Improvements

The general expression derived by Rao *et al.* (1970) for calculating the second vertical derivative of gravity $g_{zz}(h)$ from the observed Bouguer gravity data using Peters' approach and Elkins' approach, can be written as

$$h^2\left(g_{zz}(h)\right) = C_n\,\bar{g}(r_n) \tag{7}$$

with

$$\sum_{n=o}^{N} C_n = O$$

For the sake of comparison, we must calculate the amplitude response functions of equation (7). To calculate the Fourier transform of the average gravity value $\bar{g}(r)$ over a circle of radius r, two different approaches are in use: (a) an infinite number of points (Mesko 1965) and (b) a finite number of points (Swartz 1954). Because of technical difficulties, we will use in this work the infinite number of points average approach, as described by Mesko (1965).

The amplitude responses of the derived coefficient sets proposed previously by Rao *et al.* (1970) and their simple formula coefficient set are computed and the corresponding amplitude responses, along with the theoretical response of the second derivative operation are shown in Figures 1-5.

The frequency responses of the coefficient sets (Figs. 1-5) reveal:

(1) The superiority of the coefficient sets derived following Peters' approach over those sets derived following Elkins' approach when using the same ring system.

(2) The superiority of the coefficient sets derived with a weightage of $1/r^4$ to all circles over those derived with a weightage of $1/r^2$ or without any weightage to all circles.

(3) That many coefficient sets derived following Peters' approach give more accurate results than the simple formula. However, the superiority of the coefficient set derived in the simple formula over all coefficient sets derived following Elkins' approach is clear.

(4) The coefficient sets derived with preference to central point give better results than those derived with no preference to the central point, when using the same ring system and the same weightage to the circles.

(5) The coefficient sets derived using small ring system give more accurate results than those sets derived using large ring system when using the same approach and the same weightage to all circles.

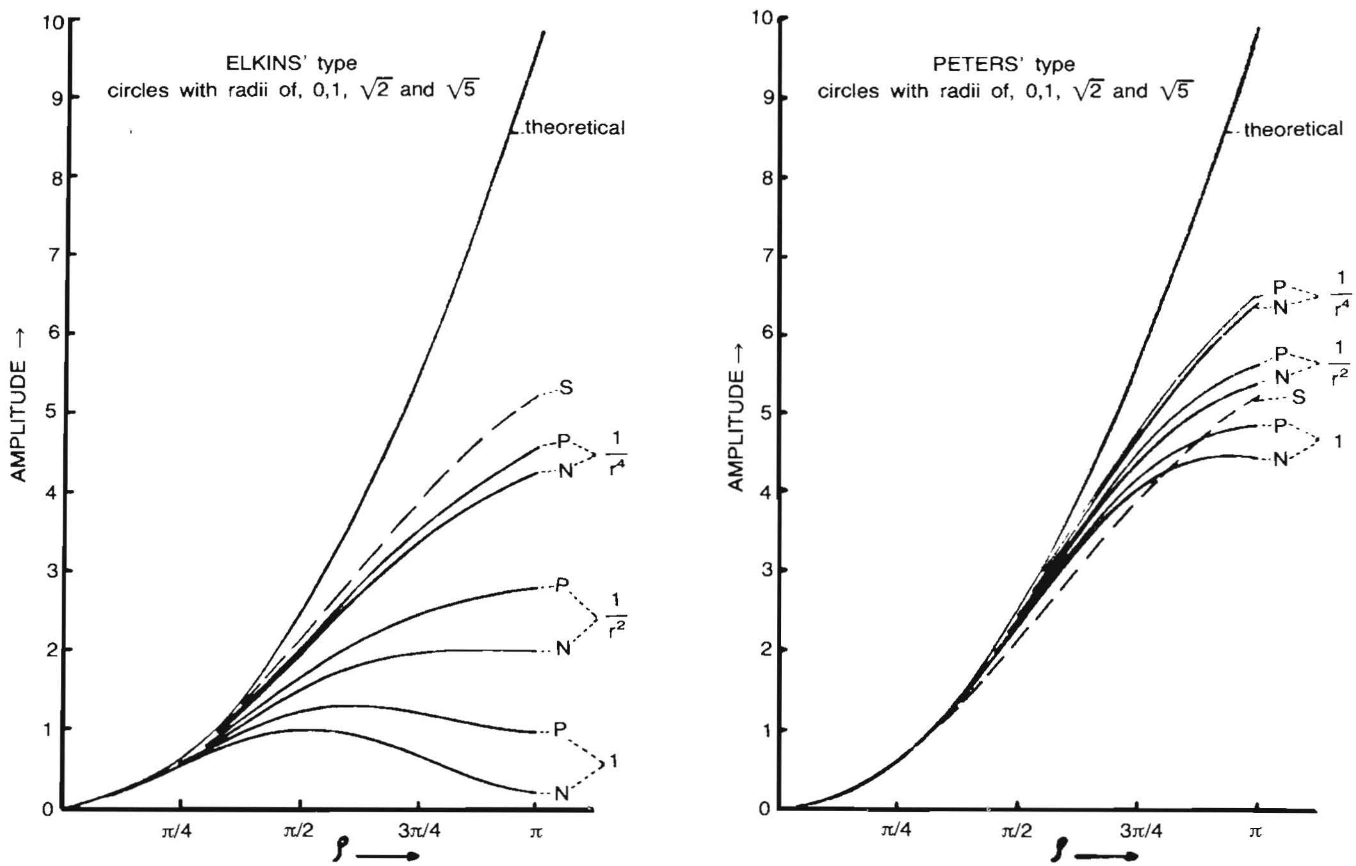

Fig. 1. Amplitude responses of previously proposed set of weights. The grid system consists of the central point and the circles of radii 1, $\sqrt{2}$, and $\sqrt{5}$.

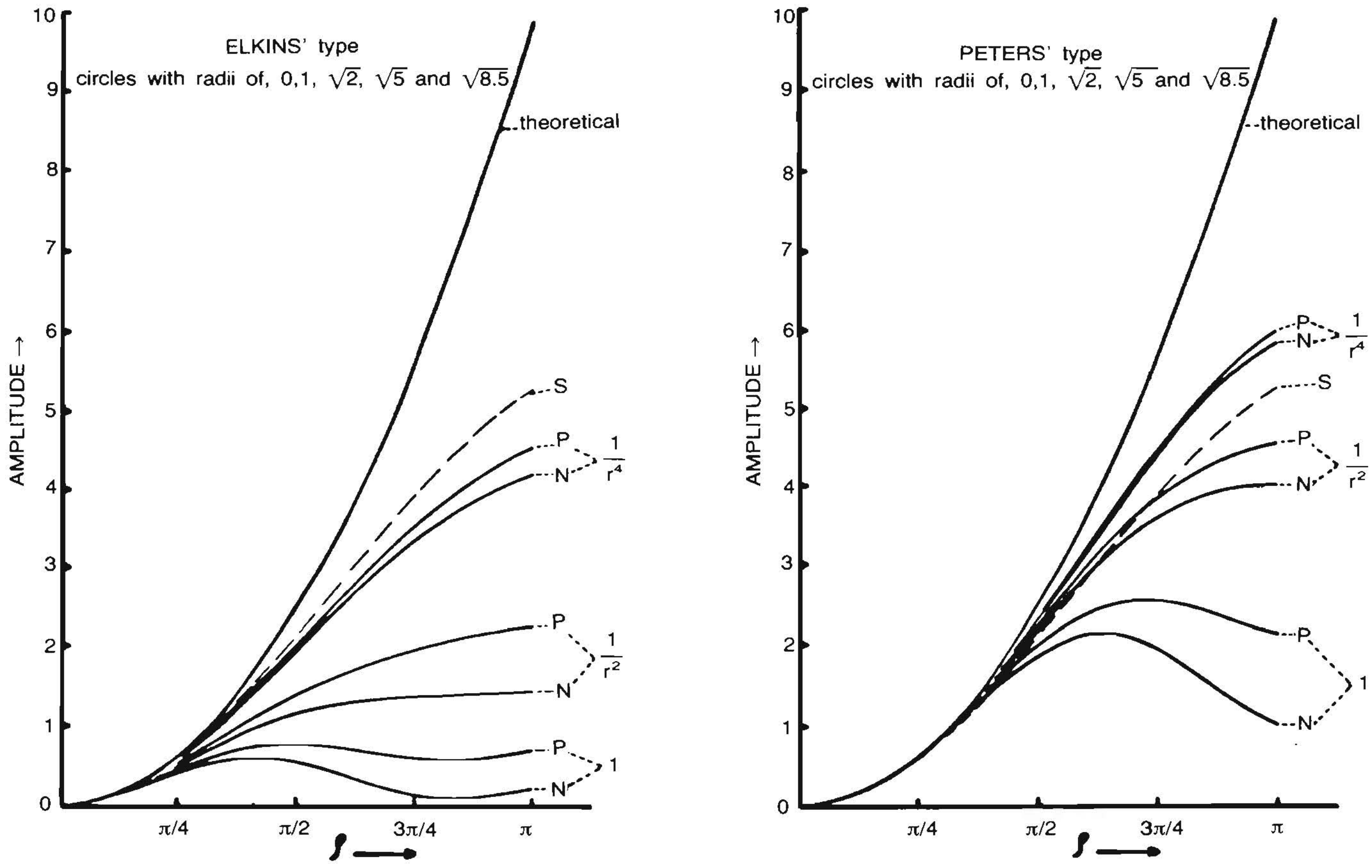

Fig. 2. Amplitude responses of previously proposed set of weights. The grid system consists of the central point and the circles of radii 1, $\sqrt{2}$, and $\sqrt{5}$, and $\sqrt{8.5}$.

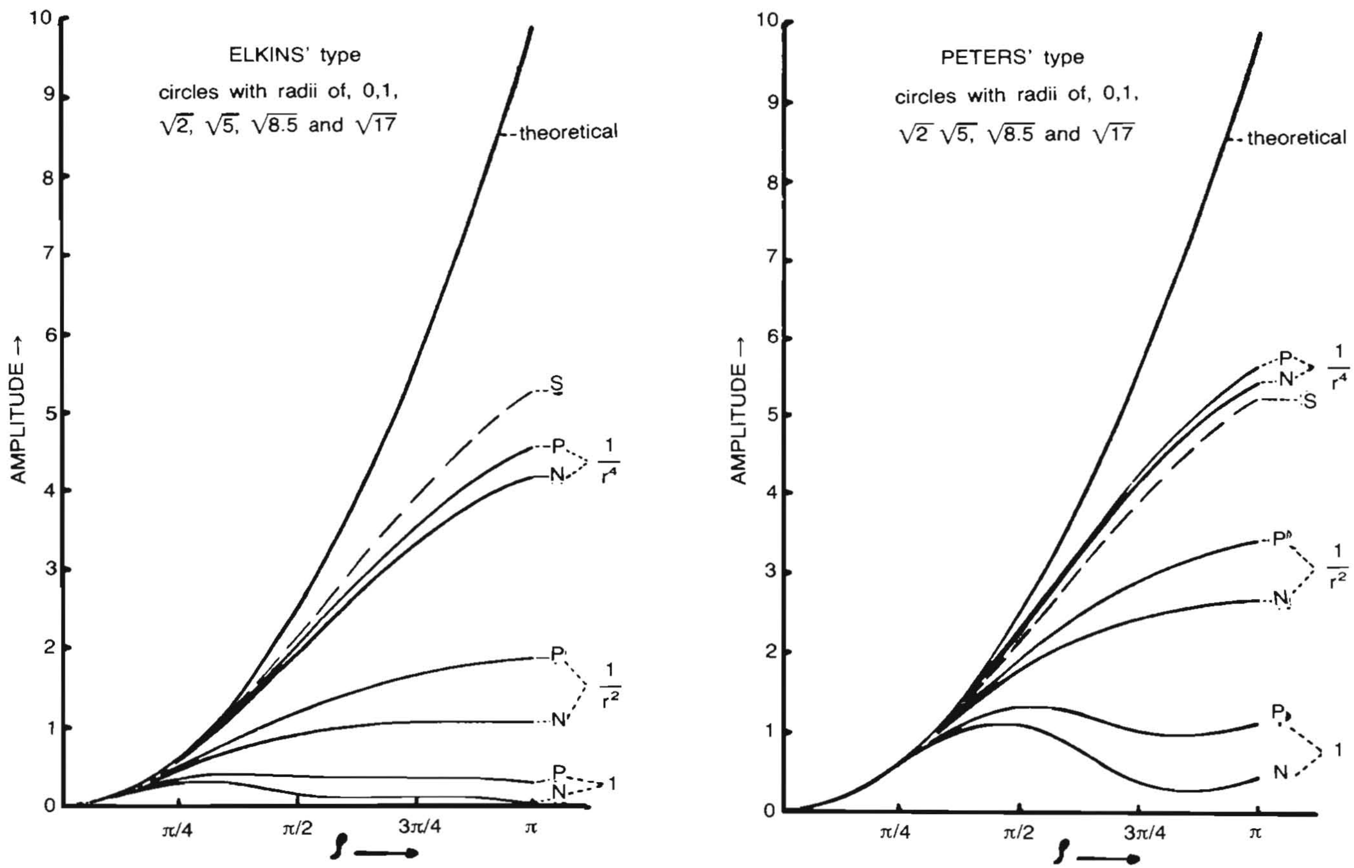

Fig. 3. Amplitude responses of previously proposed set of weights. The grid system consists of the central point and the circles of radii 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8.5}$, and $\sqrt{17}$.

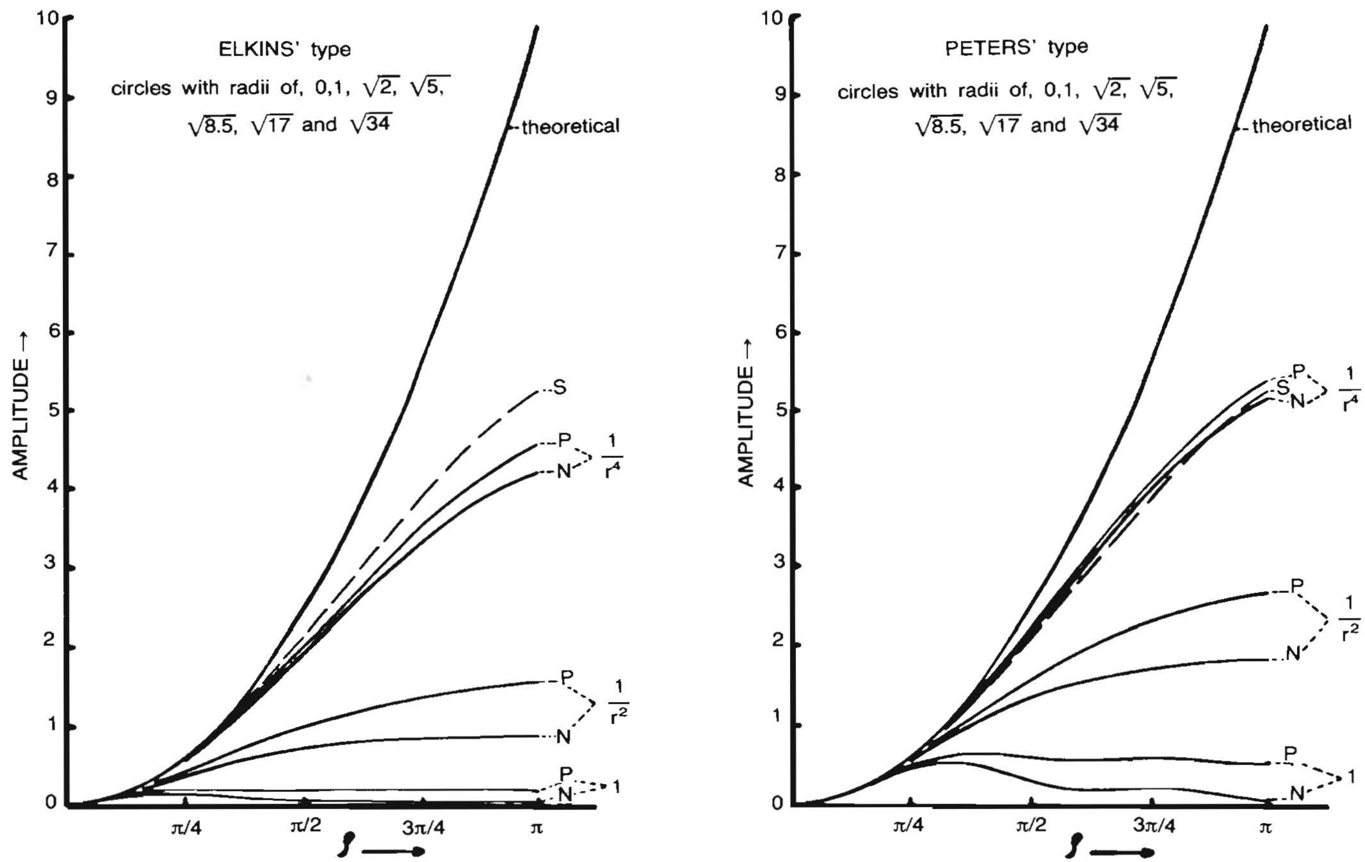

Fig. 4. Amplitude responses of previously proposed set of weights. The grid system consists of the central point and the circles of radii 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8.5}$, , $\sqrt{17}$, and $\sqrt{34}$.

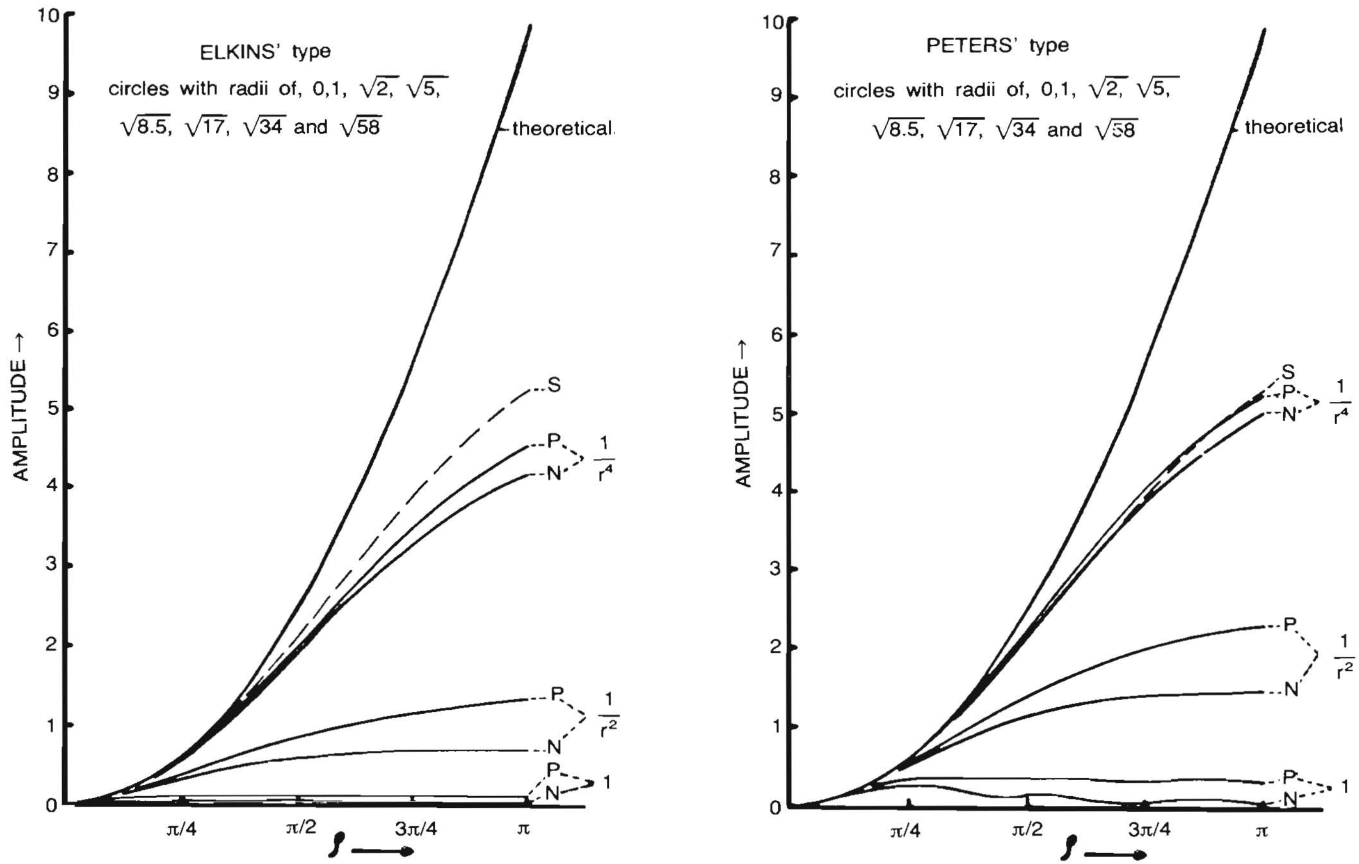

Fig. 5. Amplitude responses of previously proposed set of weights. The grid system consists of the central point and the circles of radii 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8.5}$, , $\sqrt{17}$, $\sqrt{34}$, and 58.

It thus evident that the coefficient set derived following Peters' approach with a weightage of $1/r^4$ to the circles, 1, $\sqrt{2}$, and $\sqrt{5}$ and with preference to the central point is the best among the other coefficient sets including also the coefficient set of the simple formul. It is also clear that Peters' method (1949) and Elkins' method (1951) are not completely objective because the results depend to a considerable degree on the operator's judgement in deciding which weightages to the cirlces to use.

However, for the calculation of the second derivative, new weight coefficient sets which use the least possible number of circles for obtaining average values and at the same time yield good results can be developed by making use of Richardson's improvement formula of the derivatives (Maron 1982).

Let the plane of observation be horizontal and the gravity anomaly g(x,y,z) be continuous and infinitely differentiable at all the points of the free space z = O. Let g(x,y,z) satisfy Laplace's equation

$$g_{xx} + g_{yy} + g_{zz} = O, \tag{8}$$

or

$$g_{zz} = -(g_{xx} + g_{yy}), \tag{9}$$

which provides a simple criterion to calculate the second vertical derivative of a three-dimensional feature from the observed gravity data.

Let P(x,y) be the point at which the second vertical derivative is to be computed. Let us cover the whole area with a mesh of square grids of spacing h, with P(x,y) as one of the corners of the map over a three-dimensional body. Let g(x,y+h), g(x,y−h), g(x+h,y), g(x−h,y), and g(x,y) be the anomalies at five points over the body (Fig. 6). Then using Laplace's equation and by making use of

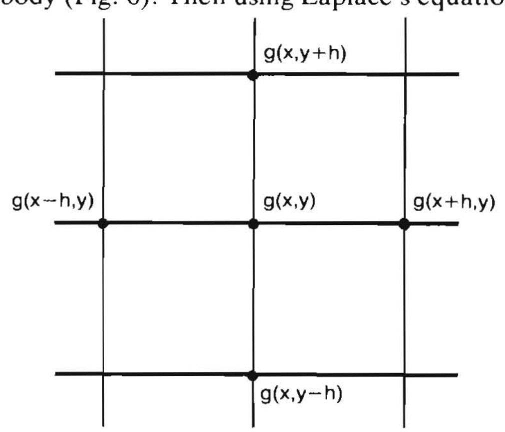

Fig. 6. A square grid with spacing h. g(x,y+h), g(x+h,y), g(x,y−h), g(x−h,y), and g(x,y) are the gravity anomalies at five points.

Taylor's formula, we can prove that

$$g_{zz} = \left(4\ \bar{g}(O) - 4\ \bar{g}(h)\right)/h^2 + O(h^2), \tag{10}$$

or

$$g_{zz} = \left(g_{zz}(h)\right)_o + O(h^2) \tag{11}$$

Since $\left(g_{zz}(h)\right)_o$ is an $O(h^2)$ approximation of the exact second derivative g_{zz}, we use it to get two approximations $g_{zz}(h)_o$ and $g_{zz}(h_{larger})$. Then an improved approximation of g_{zz} is given by Richardson's formula as

$$\left(g_{zz}(h)\right)_1 = \frac{q^2\left(g_{zz}(h)\right)_o - \left(g_{zz}(h_{larger})\right)_o}{q^2 - 1}, \tag{12}$$

where

$$q = h_{larger}/h$$

The $\left(g_{zz}(h)\right)_1$ is now an $O(h^4)$ approximation of the exact second derivative g_{zz}, that is

$$g_{zz} = \left(g_{zz}(h)\right)_1 + O(h^4) \tag{13}$$

In this case we can use again Richardson's formula to get still higher-order approximation which is more accurate than those given in (10) and (12),

$$\left(g_{zz}(h)\right)_2 = \frac{q^4\left(g_{zz}(h)\right)_1 - \left(g_{zz}(h_{larger})\right)_1}{q^4 - 1}, \tag{14}$$

where again

$$q = h_{larger}/h$$

and so on.

Then for practical use, (12) and (14) with $q = 2$, and $h_{larger} = 2h$, give

$$\begin{aligned} h^2\left(g_{zz}(h)\right)_1 = {} & 5.00000000\ g(O) \\ & -5.33333333\ \bar{g}(h) \\ & +0.33333333\ \bar{g}(2h) \end{aligned} \tag{15}$$

and

$$h^2(g_{zz(}h))_2 = \begin{aligned} &5.25000000\ g(O) \\ &-5.68888889\ \bar{g}(h) \\ &+0.44444444\ \bar{g}(2h) \\ &-0.00555555\ \bar{g}(4h) \end{aligned} \qquad (16)$$

for the first and the second improvements of the second derivative of gravity operation, respectively. The amplitude responses of equations (10, (15), and (16) along with that of the best set derived by Rao *et al.* (1970) are shown in Fig. 7, using an infinite number of points average approach.

Discussion of the Results

It can be seen that the amplitude responses of the coefficient sets developed by the help of Taylor series and Richardson's improvement formula (S $O(h^4)$ and S $O(h^6)$, Fig. 7) are close to the theoretical amplitude response of the second derivative operation. It is clear that the amplitude response of the first improved approximation set (S $O(h^4)$) is better than the amplitude response of the coefficient set derived from the simple formula (S $O(h^2)$), and the amplitude response of the second improved approximation set (S $O(h^6)$) is the best among them. This indicates that further applications of Richardson's formula can be used to obtain more accurate large filters, if required. It is also established that, in spite of the use of the same number of rings, the Richardson's approximation coefficient sets estimate the second derivative more accurately than do Rao *et al.* approximation sets.

Conclusion

The present paper confirms the basic idea of several authors on the usefulness of frequency analysis in judging the coefficient sets of the second derivative of gravity. The second derivative coefficient sets, developed by using Taylor series and Richardson's improvement formula and those sets given by Rao *et al.* (1970), are analyzed in the frequency domain. Frequency analysis clearly reveals that Richardson's approximation sets provide an improvement, in the sense of their close fit to the theoretical second derivative response, over the previously proposed coefficient sets. Thus, it has been established that, in spite of the use of the same number of rings, the Richardson's approximation coefficient sets estimate the second derivative more accurately than do Rao *et al.* approximation sets. It is also emphasized that no rigorous calculations are used in deriving our new coefficient sets. This illustrates the desirability of the present approach compared to the other approaches (Peters 1949, Elkins 1951, Rao *et al.* 1970) which use tedious techniques while computing second derivative coefficient sets.

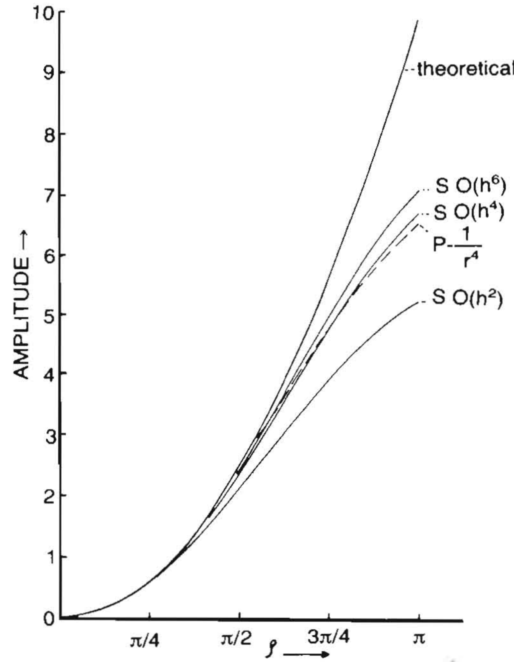

Fig. 7. A comparative picture of the frequency responses of present derived sets of weights along with the best of those of previously proposed set of weights.

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دراسة مقارنة لبعض المجموعات السابقة لمعاملات المشتقة الثانية ذات الثلاثة أبعاد باستخدام طرق استجابتها الترددية مع إضافات جديدة

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يؤكد هذا البحث على مدى جدوى استخدام طرق التحليل الترددي في الحكم على دقة مجموعات المعاملات الخاصة بالمشتقة الثانية الرأسية للمجال التثاقلي. وقد تمت مقارنة لبعض المجموعات السابقة لمعاملات المشتقة الثانية ذات الثلاثة أبعاد باستخدام دراسة استجابتها الترددية بالإضافة إلى تلك التي اشتقت بواسطة استخدام مفكوك تايلر وعلاقة ريشاردسون. كما أوضحت الدراسة تفوق المعاملات التي اشتقت باستخدام طريقة "بيتر" على تلك المعاملات التي اشتقت بطريقة "الكنز" وتفوق معاملات المشتقة الثانية التي اشتقت باستخدام أوزان مقدارها ١/ر ، حيث ر يمثل نصف قطر الدوائر جميعها، على المعاملات التي أعطيت أوزان مقدارها ١/ر٢ أو الوحدة كما تبين أيضاً أن مجموعة المعاملات التي أخذت في الاعتبار عند اشتقاقها النقطة المركزية تعطى نتائج أفضل من تلك التي لم تأخذ في الاعتبار عند اشتقاقها النقطة المركزية.

ومع ذلك أمكن باستخدام طرق التحليل الترددي الحكم على مجموعات المعاملات الخاصة بايجاد المشتقة الثانية للمجال التثاقلي والمشتقة بواسطة مفكوك تايلر ومعادلة ريشاردسون وقد أثبت أن هذه المجموعات تفوق بكثير جميع المعاملات السابقة والتي اشتقت جميعها بواسطة روى وزملائه وان هذه المعاملات تعطى نتائج أفضل بكثير.

ولقد إتضح أيضاً أن استخدام معادلة ريشاردسون في اشتقاق معاملات المشتقة الثانية يتميز بعدم استخدام طرق حسابية معقدة كما هو متبع في الطرق

السـابقة . وقـد أعطى المؤلفـون في هذا البحث مجمـوعتين جـديدتـين لمعامـلات المشتقة الثانية للمجال التثاقلي .