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On Theorems of Walsh and Laguerre Concerning Zeros of Polynomials

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ABSTRACT. We obtain a general theorem on the location of null-sets of certain types of abstract polynomials in vector spaces of arbitrary dimension (finite, or otherwise). This theorem generalizes Walsh's two-circle theorem concerning the critical points of rational functions of the form f/g, where f and g are complex-valued polynomials of the same degree; and it offers an extension of Laguerre's theorem on polar-derivatives.

1. Introduction

Let **C** represent the field of complex numbers, identified as complex plane, and let *D*(**C**) denote the family of all *classical circular regions* (briefly, *c. r.*) in **C**, *i.e.*, all open (or closed) connected subsets of the complex plane whose boundary is a circle or a straight line (including the empty set ϕ and the whole plane **C**). We denote by π_n (**C**) the class of all polynomials f: $\mathbf{C} \rightarrow \mathbf{C}$ of degree n and by Z(f) the set of all zeros of f ε π_n (**C**). Given $\zeta \varepsilon \mathbf{C}$ and a polynomial f ε π_n (**C**), we define, Marden (1966), the *polar-derivative* f(ζ,.) of f with *pole* ζ to be the polynomial

$$(1.1) \qquad f_1(z) \equiv f_1(\zeta,z) = nf(z) - (z - \zeta) f'(z).$$

The present paper rallies around the following two wellknown results of Walsh (1921) and Laguerre (1898) stated as Theorems (20,1) and (13,1) in Marden (1966).

Theorem 1.1

Let $B_i \equiv B(c_i, r_i)$ *denote the closed disk with center* c_i *and radius* r_i, $i=1,2$. *If* $f \varepsilon \pi_n$ (**C**) *and* $g \varepsilon \pi_m$(**C**) *such that* $Z(f) \subseteq B_1$ *and* $Z(g) \subseteq B_2$, *then all finite zeros of* *the derivative of* f/g *lie in the set:*

(a) $B_o \cup B_1 \cup B_2$, *where* $B_o \equiv B(c,r)$ *and*

$$c = (mc_1 - nc_2) / (m-n), \; r = (mr_1 + nr_2) / (m-n),$$

provided $m \neq n$;

(b) $B_1 \cup B_2$, provided $m = n$.

Theorem 1.2

If $f \in \pi_n(\mathbf{C})$ *and* $C \in D(\mathbf{C})$ *such that* $Z(f) \subseteq C$, *then* $Z(f_1) \subseteq C$ *for all* $\zeta \notin C$, *where* $f_1(z) \equiv f_1(\zeta,z)$ *is the polar-derivative of* f *defined by* (1.1).

Section 2 contains the most relevant details about generalized circular regions, abstract polynomials and their pseudo-derivatives in the set up of vector spaces of arbitrary dimension. These concepts are utilized in Section 3 to obtain a general result, whose complex plane versions yield Walsh's Theorem (1.1) (b) and a new result that extends Laguerre's Theorem 1.2 to a more general type of polar-derivatives. The corresponding generalization of Theorem 1.1 (a) is in Zaheer and Khan (1980).

2. Preliminaries

Throughout the paper we let $K = K_o(i) = \{z: z = a + ib; a,b \in K_o\}$, $-i^2 = 1$, represent an (arbitrary) algebraically closed field of characteristic zero, with K_o as a maximal ordered subfield of K (see Bourbaki 1952, Hörmander 1954 and Waerden 1964), so that $K_o = \mathbf{R}$, the field of reals, when $K = \mathbf{C}$. We write K_{o+} for the set of all non-negative elements of K_o. The definition of z, Re z, Im z and $|z|$ for elements $z \in K$ and the notion of K_o-convexity for subsets of K automatically come from the corresponding notions in **C** (by replacing the role of **R** by K_o). Similarly, the idea of *homographic transformations* of the projective field $K_\infty \equiv K \cup \{\infty\}$, where ∞ has the properties of scalar infinity, is an immediate extension of that of linear fractional transformations of $\mathbf{C}_\infty$. We denote by $D(K_\infty)$ the family of all *generalized circular regions* (briefly, *g. c. r.*) of K_∞, a concept originally due to Zervos (see Zervos 1960), built upon the concepts of homographic transformations and K_o- convexity (see Zaheer and Alam 1980). The sets ϕ, K, K_∞, $\{x\}$ and $K_\infty - \{x\}$, for $x \in K$, are trivial members of $D(K_\infty)$.

The following results are due to Zervos (see Zervos 1960 and Zaheer and Alam 1980).

Proposition 2.1

Every nontrivial member of $D(\mathbf{C}_\infty)$ *is the open interior (or exterior) of a circle or an open half-plane, adjoined with a connected subset (possibly empty) of its boundary. So that the open or closed member of* $D(\mathbf{C}_\infty)$, *restricted to* $\mathbf{C}$, *form the family* $D(\mathbf{C})$ *of c.r.'s in* $\mathbf{C}$.

Proposition 2.2

Every homographic transformation permutes $D(K_\infty)$.

For full details about K and $D(K_\infty)$ the reader may consult Zaheer and Alam (1980).

In the sequel, we let E denote a vector space over K of arbitrary dimension and write $E_\omega \equiv E \cup \{\omega\}$, where ω has the properties of vector infinity. Also we denote by $D^*(E_\omega)$ the family of all *supergeneralized circular regions* of E_ω as defined below, a concept introduced by Zaheer (1988).

Definition 2.3

Given $S \subseteq E_\omega$, we write

$$(2.1) \qquad G_S(x,y) = \{\rho \in K_\infty : x + \rho y \in S\} \qquad \forall\, x,y \in E.$$

We say that $S \in D^*(E_\omega)$ if $G_S(x,y) \in D(K_\infty)$ for all $x,y \in E$.

Clearly, ϕ, E, E_ω, singletons $\{x\}$ (and their complements in E_ω) are trivial members of $D^*(E_\omega)$. Since $G_S(x,0)$ is K or ϕ according as $x \in S$ or $x \notin S$ (cf. properties of ω and ∞), we have

$$\infty \in G_S(x,0) \in D(K_\infty) \qquad \forall\, x \in E$$

and

$$(2.2) \qquad \infty \in G_S(x,y) \qquad \forall\, x \in E,\ y \in E - \{0\} \Leftrightarrow \omega \in S.$$

Consequently,

$$(2.3) \qquad S \in D^*(E_\omega) \Leftrightarrow G_S(x,y) \in D(K_\infty) \qquad \forall\, x \in E,\ y \in E - \{0\}.$$

Remark 2.4

(I) In case E = K, we may use ω and ∞ interchangeably. (II) Some interesting properties and examples of nontrivial members of $D^*(E_\omega)$ have already been discussed in Zaheer (1988).

Proposition 2.5

If $S \in D(K_\omega)$ *then* $S \in D^*(K_\omega)$.

Proof. Here E = K and we write $\omega \equiv \infty$. If $S \in D(K_\omega)$ then $G_S(x,y) = f(S) \in D(K_\infty)$ for all $x,y \in E$ $(y \neq 0)$, where $f(\sigma) = (\sigma - x)/y$ (for $\sigma \in K_\infty$) is a homographic transformation of K_∞. Hence $S \in D^*(K_\omega)$ by (2.3).

The family $\pi_n(E,K)$ of all *abstract polynomials* (briefly, *a.p.*) of degree n, n $\geqslant$ 1, from E to K is defined (see Zaheer 1982, Taylor 1938, Hille and Phillips 1957) in the following way: We say that $P \in \pi_n(E,K)$ if $P: E \rightarrow K$ such that, for each $x,y \in E$,

$$(2.4) \qquad P(x + \rho y) = \sum_{k=0}^{n} A_k(x,y)\rho^k \qquad \forall\ \rho \in K,$$

where $A_k(x,y) \in K$ are independent of ρ and $A_n(x,y) \not\equiv 0$. We then define the *null-set* and the *faithful-set* of P respectively by

$$Z(P) = \{x \in E: P(x) = 0\}$$

and

$$F(P) = \{h \in E: h \neq 0;\ A_n(0,h) \neq 0\}.$$

Next, given an a.p. $P \in \pi_n(E,K)$ via (2.4) and an element $h \in F(P)$ [$F(P) \neq \phi$ as shown in Zaheer 1982, Relation (2.3)], we define the *kth pseudo-derivative* $P_h^{(k)}$ of P (relative to h) to be the mapping $P_h^{(k)}: E \rightarrow K$ given by

$$(2.5) \qquad P_h^{(k)}(x) = (k!)\ A_k(x,h) \qquad \forall\ x \in E.$$

First few members are denoted by P'_h, P''_h, etc. If $P \in \pi_n(E,K)$ is given by (2.4) and $h \in F(P)$, we know (see Zaheer 1982, Proposition 2.3 and Remark 2.4) that $P_h^{(k)} \in \pi_{n-k}(E,K)$, that

$$(2.6) \qquad h \in F(P_h^{(k)}) \text{ and } (P_h^{(k)})'_h = P_h^{(k+1)} \qquad \forall k$$

and (see Zaheer 1982, equation (2.7)) that

$$(2.7) \qquad P_h^{(k)}(x + \rho h) = \sum_{j=k}^{n} j(j-1)\cdots(j-k+1)\, A_j(x,h)\rho^{j-k}.$$

Further details (including precise references) about the above material on a.p.'s can be found in Zaheer 1982, Section 2.

The following theorem will be needed in the sequel. It simultaneously generalizes Lucas' theorem (see Marden 1966, theorem (6.1)′), Zervos' theorem (see Zervos 1960, theorem 4, p. 360) and a result due to Zaheer (see Zaheer 1982, theorem 3.4).

Theorem 2.6

(Zaheer 1988, Theorem 3.3). *If* $P \in \pi_n(E,K)$ *and* $S \in D^*(E_\omega)$ *such that* $\omega \notin S$ *and* $Z(P) \subseteq S$, *then* $Z(P_h^{(k)}) \subseteq S \;\forall\; h \in F(P)$, $k = 1,2,\ldots, n-1$.

3. Principal Results

In order to avoid unnecessary trivialities, we consider only a.p.'s of degree at least one. Two a.p.'s P,Q are called *faithful* if their faithful-sets are not disjoint, *i.e.*,

$$(3.1) \qquad F(P,Q) \equiv F(P)\cap F(Q) \neq \phi.$$

For example (cf.(2.6)) every pair from the collection $\{P, P_h', P_h'', \ldots, P_h^{(n-1)}\}$ is faithful. Other examples of such polynomials (not related to the same P) have been dealt with in Zaheer 1988. In case E = K, see Remark 3.5 (II) for another example. Given faithful a.p.'s $P \in \pi_n(E,K)$, $Q \in \pi_m(E,K)$ and a scalar $\lambda \in K - \{0\}$, we observe that $\lambda P \in \pi_n(E,K)$, $PQ \in \pi_{n+m}(E,K)$ and $P \pm Q \in \pi_N(E,K)$, where $N \leqslant \max\{m,n\}$.

Definition (3.1)

Given faithful a.p.'s $P \in \pi_n(E,K)$, $Q \in \pi_m(E,K)$ and scalars $\mu, \nu \in K - \{0\}$, we define for each $h \in F(P,Q)$ (cf. (3.1)) an a.p. $R: E \to K$ by

$$(3.2) \qquad R = \mu\, P\, Q_h' + \nu\, Q\, P_h'$$

Remark (3.2)

We observe that $R \in \pi_N(E,K)$, where $N \leqslant m + n - 1$, and that $N = m+n-1$ if and only if $\mu m + \nu n \neq 0$. This is based on the following argument: Let P be represented by (2.4) and Q by

$$(3.3) \qquad Q(x + \rho y) = \sum_{k=0}^{m} B_k(x,y)\rho^k \quad \forall \rho \varepsilon K.$$

Then (cf.(2.7)) for each h ε F(P,Q), we have

$$(3.4) \qquad Q_h^{(k)}(x + \rho h) = \sum_{j=k}^{m} j(j-1) \ldots (j-k+1)\, B_j(x,h)\rho^{j-k}$$

and

$$R(x + \rho h) = \mu P(x + \rho h)\, Q_h'(x + \rho h) + \nu\, Q(x + \rho h)\, P_h'(x+\rho h)$$

$$= \sum_{k=0}^{m+n-1} C_k(x,h)\, \rho^k, \text{ say.}$$

A simple calculation yields (cf. (2.4), (2.7), (3.3), (3.4))

$$C_{m+n-1}(x,h) = (\mu m + \nu n)\, A_n(x,h)\, B_m(x,h).$$

Since h ε F(P,Q), we conclude that $A_n(x,h) \equiv A_n(0,h) \neq 0$ and $B_m(x,h) \equiv B_m(0,h) \neq 0$ for all x ε E (see Zaheer 1982, p. 840), and that

$$C_{m+n-1}(x,h) \equiv C_{m+n-1}(0,h) \neq 0 \Leftrightarrow \mu m + \nu n \neq 0.$$

Hence,

$$R \;\varepsilon\; \pi_{m+n-1}(E,K) \Leftrightarrow \mu m + \nu n \neq 0.$$

We now state and prove the main theorem which tells us about the location of the null-set Z(R) of the a.p. R in Definition 3.1 in the case when $\mu m + \nu n = 0$ (*i.e.* when degree of R is less than $m+n-1$ (cf. Remark (3.2)). The analogous problem for R in case $\mu m + \nu n \neq 0$ (*i.e.* when $R \;\varepsilon\; \pi_{m+n-1}(E,K)$) has already been done by the authors and would appear elsewhere (see Al-Rashed and Zaheer 1989 and Zaheer and Khan 1980). The analysis and treatment therein neither apply nor carry over to the case at hand.

Theorem 3.3.

Let P,Q,R *be as in Definition* 3.1, *with* $\mu m + \nu n = 0$. *If* $S_1, S_2 \;\varepsilon\; D^*(E_\omega)$, *with* $\omega \notin S_1 \cup S_2$ *and* $S_1 \cap S_2 = \phi$, *such that* $Z(P) \subseteq S_1$ *and* $Z(Q) \subseteq S_2$, *then* $Z(R) \subseteq S_1 \cup S_2$ $\forall$ h ε F(P,Q).

Proof. On the contrary, suppose that $R(x) = 0$ for some $x \notin S_1 \cup S_2$ and for some $h \in F(P,Q)$. Then

$$\mu P(x)\, Q'_h(x) + \nu Q(x) P'_h(x) = 0.$$

Observe that $P(x)Q(x) \neq 0$ by choice of x and that $P'_h(x)\, Q'_h(x) \neq 0$ due to the fact that $Z(P'_h) \subseteq S_1$ and $Z(Q'_h) \subseteq S_2$ by Theorem 2.6. Therefore

$$\text{(3.5)} \qquad \mu Q'_h(x) / Q(x) + \nu P'_h(x) / P(x) = 0.$$

If P and Q are given by (2.4) and (3.3) then (since K is algebraically closed) we may write

$$P(x + \rho h) = \sum_{k=0}^{n} A_k \rho^k = A_n \prod_{j=1}^{n} (\rho - \rho_j) \qquad \forall\, \rho \in K,$$

$$Q(x + \rho h) = \sum_{k=0}^{m} B_k \rho^k = B_m \prod_{j=1}^{m} (\rho - \sigma_j) \qquad \forall\, \rho \in K,$$

where $A_k \equiv A_k(x,h)$, $B_k \equiv B_k(x,h)$, $\rho_j \equiv \rho_j(x,h)$, $\sigma_j \equiv \sigma_j(x,h)$ and (as noticed in Remark 3.2) $A_n \equiv A_n(x,h) = A_n(0,h) \neq 0$ and $B_m \equiv B_m(x,h) = B_m(0,h) \neq 0$. Put $G_i = G_{S_i}(x,h)$ for $i = 1,2$ (cf. Definition 2.3). Since $P(x + \rho_j h) = 0 = Q\,(x + \sigma_j h)$, the hypotheses on P,Q and S_i imply that $\rho_j, \sigma_j \neq 0, \infty$ (since $x, \omega \notin S_1 \cup S_2$) and that $0, \infty \notin G_i \in D(K_\infty)$ for $i = 1,2$. Obviously, $\rho_j \in G_1$ and $\sigma_j \in G_2$. It is known (see Zaheer 1982, Equation (3.1), p. 844) that

$$P'_h(x) / P(x) = -\sum_{j=1}^{n} 1/\rho_j, \quad Q'_h(x) / Q(x) = -\sum_{j=1}^{m} 1/\sigma_j.$$

Hence (3.5) gives

$$\text{(3.6)} \qquad \mu \sum_{j=1}^{m} 1/\sigma_j + \nu \sum_{j=1}^{m} 1/\rho_j = 0.$$

Since $0 \notin G_i \in D(K_\infty)$, the definition of $D(K_\infty)$ says that $\theta_0(G_i)$ is a $\mathbf{K_0}$−convex subset of K ($i = 1,2$), where $\theta_0(\rho) \neq 1/\rho$ for $\rho \in K_\infty$. Since $\theta_0(\rho_j) = 1/\rho_j \in \theta_0(G_1)$ and $\theta_0(\sigma_j) = 1/\sigma_j \in \theta_0(G_2)$, this fact yields the following:

$$(1/n) \sum_{j=1}^{n} 1/\rho_j = 1/\rho \in \boldsymbol{\theta}_0(G_1) \text{ for some } \rho \in G_1,$$

$$(1/m) \sum_{j=1}^{m} 1/\sigma_j = 1/\sigma \in \boldsymbol{\theta}_0(G_2) \text{ for some } \sigma \in G_2.$$

Note that here $\rho,\sigma \neq 0,\infty$. Now (3.6) gives $\mu m/\sigma + \nu n/\rho = 0$. But $m,n \geqslant 1$ and μ, $\nu \neq 0$ with $\mu m + \nu n = 0$. Hence $\mu m(\rho-\sigma) = 0$. That is, $\rho = \sigma \,\varepsilon G_1 \cap G_2$ and so $x + \rho h = x + \sigma h \,\varepsilon\, S_1 \cap S_2$. This contradicts the hypothesis that $S_1 \cap S_2 = \phi$. The proof is now complete.

Given faithful a.p.'s $P \,\varepsilon\, \pi_n(E,K)$ and $Q \,\varepsilon \pi_m(E,K)$, we define the *formal pseudo-derivative* (relative to h) of the quotient P/Q by

$$(P/Q)'_h = (QP'_h - PQ'_h)/Q^2 \qquad \forall\, h \,\varepsilon\, F(P,Q).$$

The domain of P/Q being $E-Z(Q)$. The zeros of $QP'_h - PQ'_h$ which are not the zeros of Q will be termed as the *finite zeros* of $(P/Q)'_h$. In Theorem 3.3 if we take $-\mu = \nu = 1$ and $m = n$, then $R = QP'_h - PQ'_h$ and we get the following result.

Corollary 3.4

Let $P,Q, \pi_n(E,K)$ *be faithful. If* S_1, $S_2 \varepsilon D^*(E_\omega)$, *with* $\omega \not\varepsilon S_1 \cup S_2$ *and* $S_1 \cap S_2 = \phi$, *such that* $Z(P) \subseteq S_1$ *and* $Z(Q) \subseteq S_2$, *then the finite zeros of the formal pseudo-derivative of the quotient* P/Q (*relative to* h) *lie in* $S_1 \cup S_2$ *for all* $h \,\varepsilon\, F(P,Q)$.

In order to obtain the field-analogues of the above results, we explain some notations and terminology. Let $\pi_n(K)$, $n \geqslant 1$, be the class of all nth degree polynomials $f: K \rightarrow K$, given by

$$f(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \,\varepsilon\, K \text{ and } a_n \neq 0,$$

and denote by f' the *formal derivative* of f (see Zaheer 1982, p. 842). Given $f \,\varepsilon\, \pi_n(K)$ and $g \,\varepsilon\, \pi_m(K)$, define the *type*-m *polar-derivative* f_g of f (relative to g) by

$$f_g(z) = n\, f(z)\, g'(z) - m g(z)\, f'(z). \tag{3.7}$$

Note that the polar-derivative $f_1(\zeta,z)$ of f (cf. (1.1)) is essentially a type-1 polar-derivative of f when $g(z) = z-\zeta$.

Remark 3.5

When $E = K$, we record the following facts:

(I) As discussed in Zaheer 1982, Remark 2.4 (III), we see that $\pi_n(K) = \pi_n(K,K)$ for $n \geqslant 1$, that $F(f) = K-\{0\}$ for all $f \,\varepsilon\, \pi_n(K)$, and that $f'_h(z) = hf'(z)$ for all $z \varepsilon K$, $h \varepsilon K-\{0\}$ and $f \varepsilon \pi_n(K)$. That is, when $h = 1$, the pseudo-derivative f'_h becomes the formal derivative f'. Furthermore, when $K = \mathbf{C}$, it coincides with the familiar derivative in calculus.

(II) By the above remark

$$F(f,g) = F(f) \cap F(g) = K-\{0\} \neq \phi$$

for all $f \varepsilon \pi_n(K)$ and $g \varepsilon \pi_m(K)$. That is, every pair of polynomials from K to K is faithful.

(III) If $\mu = n$, $\nu = -m$, $f \varepsilon \pi_n(K)$ and $g \varepsilon \pi_m(K)$, we see (cf. Remarks (I) and (II)) that the polynomial R of Theorem 3.3, with $P = f$, $Q = g$ and $h = 1$, is given by

$$R = nfg' - mgf' = f_g$$

That is, R becomes a type-m polar-derivative of f. Furthermore, when $g(z) = z - \zeta$ ($\zeta \varepsilon K$), R becomes the polar-derivative of f.

(IV) Similarly, if $f,g \varepsilon \pi_n(K)$, the formal pseudo-derivative of P/Q (with $P = f$, $Q = g$ and $h = 1$) in corollary 3.4 is given by

$$(P/Q)'_h = (gf' - fg')/g^2 = (f/g)'$$

That is, it coincides with the formal derivative of f/g and, for $K = \mathbf{C}$, it reduces to the usual derivative of f/g via calculus.

(V) Let $B \equiv B(c,r)$ denote the *closed ball* in K with center $c \varepsilon K$ and radius $r \varepsilon K_{0+}$. Obviously, $\infty \notin B$. It is known (see Zaheer and Alam 1980, p. 116) that $B \varepsilon D(K_\infty)$. Consequently, $B \varepsilon D^*(K_\omega)$, $\omega \notin B$, by Remark 2.4(I) and *Proposition* 2.5.

In view of the above remarks, we deduce the following results.

Theorem 3.6

Let $f,g \varepsilon \pi_n(K)$ *and* $C_1, C_2 \varepsilon D(K_\omega)$ *such that* $\omega \notin C_1 \cup C_2$ *and* $C_1 \cap C_2 = \phi$. *If* $Z(f) \subseteq C_1$ *and* $Z(g) \subseteq C_2$, *then the finite zeros of the formal derivative of* f/g *lie* in $C_1 \cup C_2$. *In particular,* C_1 *and* C_2 *may be taken as closed balls in* K.

Proof. The proof follows from Proposition 2.5, Corollary 3.4 and Remarks 3.5 (I), (IV) and (V).

For $K = \mathbf{C}$, the above theorem furnishes an improved form of Walsh's Theorem 1.1 (b), in the sense that closed disks form a proper subfamily of $D(\mathbf{C}_\omega)$ as seen in Proposition 2.1.

Theorem 3.7

Given $f \in \pi_n(K)$, $g \in \pi_m(K)$, *let* $f_g(z)$ *be the type-*m *polar-derivative of* f *relative to* g *defined by* (3.7). *If* $C_1, C_2 \in D(K_\omega)$, *with* $\omega \notin C_1 \cup C_2$ *and* $C_1 \cap C_2 = \phi$, *such that* $Z(f) \subseteq C_1$ *and* $Z(g) \subseteq C_2$, *then*

$$Z(f_g) \subseteq C_1 \cup C_2$$

Proof. Theorem 3.3, Proposition 2.5 and Remark 3.5 (III) combine to yield the desired result.

The last theorem provides a new result on the zeros of type-m polar-derivatives of polynomials $f \in \pi_n(K)$ for $m \geqslant 1$, whereas Laguerre's Theorem 1.2 as well as its generalization to the field K (see Zervos 1960, Corollary 2.8) deals with type-1 polar-derivatives only.

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عن نظريتين لوالش ولاقير بخصوص أصفار كثيرات الحدود

عبد الله محمد الراشد و نعمت ظهير

قسم الرياضيات ـ كلية العلوم ـ جامعة الملك سعود
ص . ب (٢٤٥٥) الرياض ١١٤٥١ ـ المملكة العربية السعودية

نرمز ـ في هذا البحث ـ بـِ C لحقل الأعداد المركبـة وبـِ D (C) لعائلة المنـاطق الـدائريـة التقليديـة وهي تلك المجموعـات الجزئيـة من المستوى C (بمـا في ذلك المجموعة الخالية ϕ والمستـوى C بكامله) وتكـون مفتوحـة (أو مغلقة) ومـترابطة إضافةً إلى أن حدودها عبارة عن دوائر أو خطوط مستقيمة. نرمز كذلك بـِ π_n (C) لصف جميـع كثـيرات الحـدود $f: C \rightarrow C$ من الـدرجـة n ، وبـِ Z (f) للمجمـوعـة الصفرية للدَّالة $f \varepsilon \pi_n (C)$.

تُعرَّف المشتقة القطبية، $f_1 (\zeta , .)$، للدَّالة $f \varepsilon \pi_n (C)$ عند القطب ζ بأنها كثـيرة الحدود التالية:

$$f_1 (z) \equiv f_1 (\zeta , z) = n f (z) - (z - \zeta) f' (z)$$

يعنىٰ بحثنـا بتعميم النظريتـين التـاليتـين: الأولى وهي نـظريـة الـدائـرتـين لوالش، وتهتم بالنقـاط الحرجـة للدّوال النسبية عـلى شكل f/g حيث f و g كثـيرتا حـدود مركبتـين من نفس الدرجـة. الثانيـة وهي نـظريـة لاقـير المشهـورة وتهتم بالمشتقات القطبية.

النظرية الأولى (والش):

ليكن $B_i \equiv B (c_i , r_i)$ القرص المغلق الذي مـركزه c_i ونصف قـطره r_i، حيث $i = 1 , 2$. إذا كانت الدَّالتان $f, g \varepsilon \pi_n (C)$، بحيث $Z (f) \subseteq B_1$ و $Z (g) \subseteq B_2$ ، فإن جميع الاصفار المنتهية لمشتقة الدَّالة f/g تنتمى للمجموعة $B_1 \cup B_2$.

النظرية الثانية (لاقير):

إذا كانت الدَّالة $f \varepsilon \pi_n (C)$ وكان $E \varepsilon \mathcal{D} (C)$، بحيث $Z (f) \subseteq E$، فإن $Z (f_1) \subseteq E$ لجميع $\zeta \notin C$، حيث أن $f_1 (z) \equiv f_1 (\zeta , z)$ هي المشتقة القطبية للدَّالة f حسب التعريف المعطى سابقاً.

تدرس النتيجة الرئيسة موضوع المجموعات الصفرية لأنواع معينة من كثيرات الحدود التجريدية في الفضاءات المتجهة ذات الأبعاد الاختيارية (منتهية أو غير منتهية) معممة بذلك كلا النظريتين أعلاه. على وجه التحديد، ليكن K حقلاً جبرياً مغلقاً و E فضاءً متجهاً، ذا بعدٍ اختياري، فوق K، ولنأخذ كثيرتي الحدود التجريديتين $p \varepsilon \pi_n (E , K)$ و $Q \varepsilon \pi_m (E , K)$ والمعاملين $\mu, \nu \in K - \{ 0 \}$. نعرِّف، من أجل كل h في المجموعة المتميزة $F (P , Q)$ كثيرة حدود تجريدية $R : E \rightarrow K$ على النحو التالي:

$$R = \mu P Q'_h + \nu Q P'_h$$

إذا كان $\mu m + \nu n = 0$ وكان S_2 , S_1 ينتميان لمجموعة المناطق الدائرية الفائقة التعميم $D^* (E_W)$ حيث $W \notin S_1 \cup S_2$ و $S_1 \cap S_2 = \phi$ و $Z (P) \subseteq S_1$ و $Z (Q) \subseteq S_2$، فإن $Z (R) \subseteq S_1 \cup S_2$ وهذا من أجل كل h من $F(P , Q)$.