Arab Gulf J. Scient. Res., **8** (3), pp. 1-16 (1990)

Separation Axioms, Subspaces and Product Spaces in Fuzzy Topology

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ABSTRACT. Let (X,τ) be a topological space and let $\omega(\tau)$ be the set of all lower semi-continuous functions defined from X into the closed unit interval [0,1]. We prove the following result in this paper:

If $\{(X_i,\tau_i):i\in I\}$ is a collection of topological spaces, then $\omega(\Pi\,\tau_i) = \Pi\omega(\tau_i)$.

We introduce some fuzzy separation axioms and then we study their hereditary and productive properties. We also study the relation between spaces having the fixed point property and fuzzy connected spaces.

1. Introduction

Zadeh (1965) introduced the notion of fuzzy set in X as a function λ from X into the closed unit interval [0,1]. Then a quasi-fuzzy topology on X was introduced by Chang (1968) as a collection of fuzzy sets on X, stable for arbitrary suprema and finite infima and containing the constant fuzzy sets 0 and 1. A fuzzy topology on X as introduced by Lowen (1976) is a quasi fuzzy topology which moreover contains all constant fuzzy sets. The concept of induced fuzzy topological spaces was introduced by Weiss (1975). Since then many authors have continued the investigation of such spaces and much attention has been given to the extension of the separation notions to fuzzy topological spaces.

Let λ, μ be two fuzzy sets in X, *i.e.* $\lambda, \mu \in [0,1]^X$. We write $\lambda \subseteq \mu$ iff $\lambda(x) \leq \mu(x)$ for all $x\in X$. By $\lambda = \mu$ we mean that $\lambda \subseteq \mu$ and $\mu \subseteq \lambda$; *i.e.*, $\lambda(x) = \mu(x)$ for all $x\in X$. If $\{\lambda_i : i\in I\}$ is a collection of fuzzy sets in X, then we define

$$(\cup\lambda_i)(x) = \sup\{\lambda_i(x) : i\in I\},\ x\in X;\ \text{and}$$
$$(\cap\lambda_i)(x) = \inf\{\lambda_i(x) : i\in I\},\ x\in X.$$

Let r denote the fuzzy set given by $r(x) = r$ for all $x \in X$, where $0 \leq r \leq 1$; *i.e.* r denotes the "constant" fuzzy set at level r. The complement λ' of a fuzzy set λ on X is given by $\lambda'(x) = 1 - \lambda(x)$, $x \in X$. If $A \subseteq X$, then χ_A denotes the characteristic function of A. A fuzzy set λ is called crisp iff $\lambda(x) \in \{0,1\}$ for all $x \in X$. If (X,τ) is an ordinary topological space then two quasi-fuzzy topologies on X are associated with τ, namely:

(i) the class of all lower semicontinuous functions between (X,τ) and $([0,1], \tau_u)$ with τ_u the usual topology on $[0,1]$. This fuzzy topology is denoted by $\omega(\tau)$.

(ii) the class of all characteristic functions of τ-open sets in X. This quasi-fuzzy topology is denoted by $X|\tau$.

The closure $c1\lambda$ (or $\overline{\lambda}$) and the interior int λ (or λ^o) of a fuzzy set λ in a (quasi) fuzzy topological space (we write (q)fts for short) (x,Γ) are defined by

$$\overline{\lambda} = \cap \{\mu : \mu' \in \Gamma \text{ and } \lambda \subseteq \mu\}$$

$$\lambda^o = \cup \{\mu : \mu \in \Gamma \text{ and } \mu \subseteq \mu\}.$$

In this paper we shall follow Wong (1974) for the definitions of : fuzzy point, a basis and a subbasis for a (quasi) fuzzy topological space, the direct and the inverse images of a fuzzy set, the product (quasi) fuzzy topology and fuzzy continuous mapping. For instance, a fuzzy point p in a set X is a fuzzy set in X given by $p(x) = t$ for $x = x_p$ $(0 < t < 1)$ and $p(x) = 0$ for $x \neq x_p$. $x_p \in X$ is called the support of p and $p(x_p) = t$ the value (level) of p. However, we shall agree that a fuzzy crisp point q in X is a fuzzy set in X given by $q(x) = 1$ for $x = x_q$ and $q(x) = 0$ for $x \neq x_q$. We shall follow Srivastava, *et al.* (1981) for the definition of 'belonging to'. Namely: A fuzzy point p in X is said to belong to fuzzy set λ in X (notation: $p \in \lambda$) iff $p(x_p) < \lambda(x_p)$. Finally, two fuzzy points p and q are said to be distinct iff their supports are distinct, *i.e.*, $x_p \neq x_q$.

1.1 ***Definition***

Let $f:(X,\Gamma_1) \to (Y,\Gamma_2)$ be a function from a fts (X,Γ_1) to a fts (Y,Γ_2). The function f is a fuzzy continuous iff the inverse image of every Γ_2-open fuzzy set in Y is Γ_1-open or, equivalently, iff the inverse image of every Γ_2-closed fuzzy set in Y is Γ_1-closed. The function f is fuzzy open (fuzzy closed) iff the direct image of every (Γ_1-open) (Γ_1-closed) fuzzy set in X is Γ_2-open (Γ_2-closed). The function f is fuzzy homeomorphism iff f is bijective, continuous and open.

1.2 ***Definition***

Let (X,Γ) be a fts and $A \subseteq X$. Then the family $\Gamma_A = \{\lambda|A : \lambda \in \Gamma\}$ is a fuzzy topology on A, where $\lambda|A = \lambda \cap \chi_A$ is the restriction of λ to A. Then (A,Γ_A) is called the fuzzy subspace of the fts X with underlying set A.

It is easy to see that a fuzzy set μ in A is fuzzy closed in A iff there exists a fuzzy closed set λ in X such that $\mu = \lambda|A$.

1.3 ***Definition***

A fuzzy topological property P is called hereditary (weakly hereditary, hereditary with respect to open subspaces), iff each subspace (closed subspace, open subspace) of a fts with property P also has property P.

1.4 ***Definition***

Let $\{(X_\alpha,\Gamma_\alpha) : \alpha \in \triangle\}$ be a family of non-empty fuzzy topological spaces. Let $X = \Pi X_\alpha$ be the usual product of X_α's and P_α be the projection from X to X_α. The fuzzy topology generated by $\psi = \{P_\alpha^{-1}(\lambda_\alpha) : \lambda_\alpha \in \Gamma_\alpha, \alpha \in \triangle\}$ as subbasis is called the product fuzzy topology in X. Clearly if λ is a basic element in the product topology, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \triangle$ such that

$$\lambda(x) = \min \{\lambda_{\alpha_i}(x_{\alpha_i}) : i = 1,2, \ldots,n\} \text{ for each } x = (x_\alpha)_{\alpha \in \triangle} \in X.$$

The fuzzy product topology of $(X_\alpha, \Gamma_\alpha)$, $\alpha \in \triangle$, will be denoted by $(\Pi X_\alpha, \Pi \Gamma_\alpha)$. If λ_α is a fuzzy set in X_α and $x = (x_\alpha)_{\alpha \in \triangle} \in X$ then we define

$$(\Pi\lambda_\alpha)\ (x) = \inf \{\lambda_\alpha(x_\alpha) : \alpha \in \triangle\}.$$

It is clear that the collection of all fuzzy sets of the form $\Pi\lambda_\alpha$; where $\lambda_\alpha \in \Gamma_\alpha$ for each $\alpha \in \triangle$ and for all but finitely many coordinates $\lambda_\alpha = 1_\alpha$; forms a base for the fuzzy product topology..

Now we prove the following elementary results as we shall use them in the sequel.

1.5 *Lemma*

Let f: $(X,\Gamma_1) \rightarrow (Y,\Gamma_2)$ be a function and μ a fuzzy set in Y. Then $(f^{-1}(\mu))^{-1}(A) = f^{-1}(\mu^{-1}(A))$ where $A \subseteq [0,1]$.

Proof

Straightforward.

1.6 *Theorem*

Let (X,τ_1) and (Y,τ_2) be two topological spaces. Then $f : (X,\tau_1) \to (Y,\tau_2)$ is continuous if and only if $f : (X,\omega(\tau_1)) \to (Y,\omega(\tau_2))$ is fuzzy continuous.

Proof

$(\to)$ Let $\mu \in \omega(\tau_2)$ and $a \in [0,1]$. Then $(f^{-1}(\mu))^{-1}\,(a,1] = f^{-1}(\mu^{-1}(a,1]) \in \tau$ because $f : (X,\tau_1) \to (Y,\tau_2)$ is continuous and $\mu^{-1}(a,1] \in \tau_2$.

$(\leftarrow)$ Let $U \in \tau_2$. Then $\chi_U \in \omega(\tau_2)$ and hence

$f^{-1}(U) = f^{-1}\,((\chi_U)^{-1}(\frac{1}{2},1]) = (f^{-1}\chi_U))^{-1}(\frac{1}{2},1] \in \tau_1$.

1.7 *Theorem*

Let (X,τ_1) and (Y,τ_2) be two topological spaces. Then $f : (X,\tau_1) \to (Y,\tau_2)$ is continuous if and only if $f : (X,X|\tau_1) \to (Y,Y|\tau_2)$ is fuzzy continuous.

Proof

Straightforward.

1.8 *Theorem*

Let (X,τ) be a topological space and $A \subseteq X$. Then $\omega(\tau_A) = \omega(\tau)_A$ provided that $A \in \tau$.

Proof

Let $\lambda \in \omega(\tau_A)$. Then $\lambda : (A,\tau_A) \to [0,1]$ is a lower semicontinuous function. Define $\mu : (X,\tau) \to [0,1]$ by $\mu(x) = \lambda(x)$ if $x \in A$ and $\mu(x) = 0$ if $x \in X\text{-}A$. Then $\mu^{-1}\,(a,1)] = \lambda^{-1}\,(a,1] \in \tau_A \subseteq \tau$ for all $a \in [0,1]$. Therefore $\mu \in \omega(\tau)$. Notice that $\lambda \in \omega(\tau)_A$ because $\lambda = \mu \cap \chi_A$. Consequently, we have proved that $\omega(\tau_A) \subseteq \omega(\tau)_A$. To prove the other inclusion, let $\lambda \in \omega(\tau)_A$. Then there exists $\mu \in \omega(\tau)$ such that $\lambda = \mu \cap \chi_A$. Notice that if $x \notin A$, then $\lambda(x) = 0$. So we may regard λ as a function from (A,τ_A) into $[0,1]$. Notice that for any $a \in [0,1]$ we have $\lambda^{-1}\,(a,1] = \mu^{-1}\,(a,1)] \cap A \in \tau_A$. Consequently $\lambda \in \omega(\tau_A)$.

2. Fuzzy Separation Axioms

Several authors have introduced different definitions of separation properties for fuzzy topologies (see, *e.g.* Hutton (1975) and (1977), Hutton and Reilly

(1980), Ming and Ming (1980a) and Srivastava, *et al.* (1981)). The Hausdorff axiom has had a hard life in fuzzy set theory since many authors have proposed different definitions (*e.g.* Ming and Ming (1980a), Sarkar (1981) and Wong (1974)).

2.1 ***Definition***

A (quasi) fuzzy topological space (X,Γ) is said to be

(1) T_0 iff for any two distinct fuzzy points p,q in X, there exists an open fuzzy set μ such that ($p\in\mu$ and $\mu \cap q = 0$) or ($q\in\mu$ and $\mu \cap p = 0$).

(2) $T_{0\omega}$ iff for any two distinct fuzzy points p, q in X, there exists an open fuzzy set μ such that $p\in\mu \subseteq q'$ or $q\in\mu \subseteq p'$.

(3) T_1 iff for any two distinct fuzzy points p, q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in \mu_1$, $\mu_1 \cap q = 0$, $q\in \mu_2$ and $\mu_2 \cap p = 0$.

(4) $T_{1\omega}$ iff for any two distinct fuzzy points p, q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in \mu_1 \subseteq q'$ and $q\in \mu_2 \subseteq p'$.

2.2 ***Definition***

A (quasi) fuzzy topological space (X,Γ) is said to be

(1) T_2 iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in \mu_1$, $q\in \mu_2$ and $\mu_1 \cap \mu_2 = 0$.

(2) $T_{2\omega}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in \mu_1$, $q\in \mu_2$ and $\mu_1 \subseteq \mu_2'$.

(3) $T_{2\frac{1}{2}}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in \mu_1$, $q\in \mu_2$ and $\bar{\mu}_1 \cap \bar{\mu}_2 = 0$.

(4) $T_{2\frac{1}{2}}$ iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in \mu_1$, $q\in \mu_2$ and $\bar{\mu}_1 \cap \bar{\mu}_2'$.

We proved in Fora (1989) (see Theorem 2.3) that our $T_{0\omega}$, $T_{1\omega}$ concepts coincide with the F T_0, F T_1 concepts, respectively, of Ghanim, Kerre and Mashhour (1984). However, our $T_{2\omega}$ concept is different from the F T_2 concept of the same preceding paper..

2.3 ***Definition***

A (quasi) fuzzy topological space (X,Γ) is said to be

(1) T_s iff all fuzzy points and fuzzy crisp "points" are closed in X.

(2) T_c iff all fuzzy crisp "points" are closed in X.

2.4 *Theorem*

Let $f : (X,\Gamma_1) \to (Y,\Gamma_2)$ be a bijective fuzzy continuous map. If (Y,Γ_2) is a T_i-space then (X,Γ_1) is a T_i-space, $i \in \{0,1,2,2\frac{1}{2},c,s,0\omega,1\omega,2\omega,2\frac{1}{2}\omega\}$.

Proof

Let p,q be two distinct fuzzy points in X. Then x_p, x_q are two distinct elements in X. Therefore $f(x_p)$, $f(x_q)$ are two distinct elements in Y. Now, let p_1, q_1 be two fuzzy points in Y determined by $p_1(f(x_p)) = p(x_p)$ and $q_1(f(x_q)) = q(x_q)$. Then p_1, q_1 are two distinct fuzzy points in the T_i - space Y. Thus there exist Γ_2-open sets λ_1,λ_2 satisfying the appropriate definition of Y being a T_i-space. Since f is fuzzy continuous, therefore $f^{-1}(\lambda_i) \in \Gamma_i$ for i = 1,2. Now, it is clear that $f^{-1}(\lambda_1)$, $f^{-1}(\lambda_2)$ are Γ_1-open sets satisfying the appropriate definition of X being a T_i-space.

2.5 ***Corollary***

"Being a T_i-space" is a fuzzy topological property for each $i \in \{0,1,2,2\frac{1}{2},c,s,0\omega,1\omega,2\omega,2\frac{1}{2}\omega\}$.

2.6 ***Definition***

A (quasi) fuzzy topological space (X,Γ) is said to be

(1) regular iff for every fuzzy point p in X and every closed fuzzy set λ in X such that $p \in \lambda'$, there exist open fuzzy sets μ_1 and $\mu_2$such that $p \in \mu_1$, $\lambda \subseteq \mu_2$ and $\mu_1 \cap \mu_2 = 0$.

(2) ω-regular iff for every fuzzy point p in X and every closed fuzzy set λ in X such that $p \in \lambda'$, there exist open fuzzy sets μ_1 and μ_2 such that $p \in \mu_1$, $\lambda \subseteq \mu_2$ and $\mu_1 \subseteq \mu_2'$.

(3) T_3 iff (X,Γ) is regular and T_c.

(4) $T_{3\omega}$ iff (X,Γ) is ω-regular and T_c.

We proved in Fora (1987) that our ω-regularity is equivalent to Hutton and Reilly's fuzzy regularity concept introduced in Hutton and Reilly (1980).

2.7 ***Definition***

A (quasi) fuzzy topological space (X,Γ) is said to be

(1) normal iff for every pair of closed fuzzy sets λ_1, λ_2 such that $\lambda_1 \subseteq \lambda_2'$, there exist open fuzzy sets μ_1,μ_2 such that $\lambda_1 \subseteq \mu_1$, $\lambda_2 \subseteq \mu_2$ and $\mu_1 \cap \mu_2 \in 0$.

(2) ω-normal iff for every pair of closed fuzzy sets λ_1, λ_2 such that $\lambda_1 \subseteq \lambda_2'$, there exist open fuzzy sets μ_1, μ_2 such that $\lambda_1 \subseteq \mu_1 \subseteq \mu_2' \subseteq \lambda_2'$.

(3) T_4 iff (X,Γ) is normal and T_s.

(4) $T_{4\omega}$ iff (X,Γ) is ω-normal and T_s.

We proved in Fora (1987) (see Theorem 3.13) that our ω-normality is equivalent to the normality concept introduced in Hutton (1975).

2.8 ***Definition*** (Hutton 1975)

The fuzzy unit interval [0,1] (L) is the set of all monotonic decreasing functions $\lambda : R \rightarrow [0,1]$ satisfying

(1) $\lambda(t) = 1$ for $t<0$ $,t \in R$,

(2) $\lambda(t) = 0$ for $t>1$ $,t \in R$;

after the identification of $\lambda:R \rightarrow [0,1]$ and $\mu:R \rightarrow [0,1]$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for every $t \in R$ (where $\lambda(t-) = \inf \{\lambda(s): s < t\}$ and $\lambda(t+) = \text{SUP} \{\lambda(s) : s > t\}$).

We define a fuzzy topology on [0,1] (L) by taking as a subbase $\{ L_t, R_t : t \in R\}$ where we define

$$L_t(\lambda) = (\lambda(t-))' \quad \text{and} \quad R_t(\lambda) = \lambda(t+).$$

This topology is called the usual fuzzy topology for [0,1] (L). It is easy to notice that $\beta = \{R_a \cap L_b : a,b \in R\}$ is indeed a base for the usual fuzzy topology on the fuzzy unit interval [0,1] (L).

2.9 ***Definition***

A (quasi) fuzzy topological space (X,Γ) is called

(1) completely regular iff for every open fuzzy set λ and every fuzzy point p in X such that $p \in \lambda$, there exists a fuzzy continuous function $f : (X,\Gamma) \rightarrow [0,1]$ (L) such that for every $x \in X : p(x) \leq f(x)\ (1-) \leq f(x)\ (0+) \leq \lambda(x)$.

(2) functionally Hausdorff (abbreviated $T_{f.h}$) iff for any two distinct fuzzy points p and q in X, there exists a fuzzy continuous function $f : (X,\Gamma) \rightarrow [0,1]$ (L) such that for every $x \in X : p(x) \leq f(x)(1-) \leq f(x)(0+) \leq q'(x)$.

(3) $T_{3\frac{1}{2}\omega}$ iff (X,Γ) is completely regular and T_s.

We proved in Fora (1989) that our concept of being a completely regular fuzzy topological space is equivalent to the definition given in Hutton (1977).

2.10 *Theorem*

(a) "Being a T_i-space" is a fuzzy topological property for all $i \in \{3, 4, f.h, 3\omega, 3\frac{1}{2}\omega, 4\omega\}$.

(b) The followings are fuzzy topological properties: regularity, normality, ω-regularity, ω-normality, complete regularity.

Proof

(a) $(i = 4\omega)$: Let $f:(X,\Gamma_1) \rightarrow (Y,\Gamma_2)$ be a fuzzy homeomorphism from a fuzzy topological space X onto a $T_{4\omega}$-space Y. Let λ_1, λ_2 be two closed sets in X such that $\lambda_1 \subseteq \lambda_2'$. Then $f(\lambda_1), f(\lambda_2)$ are closed sets in Y and moreover, we have $f(\lambda_1) \subseteq f(\lambda_2)'$ (easy calculations). Since Y is ω-normal, there exit μ_1, μ_2 open sets in Y such that $f(\lambda_1) \subseteq \mu_1$, $f(\lambda_2) \subseteq \mu_2$ and $\mu_1 \subseteq \mu_2'$. It is easy to check that $\lambda_1 \subseteq f^{-1}(\mu_1)$, $\lambda_2 \subseteq f^{-1}(\mu_2)$ and $f^{-1}(\mu_1) \subseteq (f^{-1}(\mu_2))'$. The proof is completed by noticing that $f^{-1}(\mu_1)$, $f^{-1}(\mu_2)$ are open sets in X.

The proof of the other cases is similar to the above case.

The following result is easy to prove.

2.11 *Theorem*

(i) "Being a T_i-space" is a hereditary property for $i \in \{0, 1, 2, 2\frac{1}{2}, 3, c, s, 0\omega, 1\omega, 2\omega, 2\frac{1}{2}\omega, 3\omega, 3\frac{1}{2}\omega, f.h\}$.

(ii) Regularity, ω- regularity and complete regularity are all hereditary properties.

(iii) "Being a T_j-space" is weakly hereditary for $j \in \{4, 4\omega\}$.

(iv) Normality and ω-normality are weakly hereditary properties.

2.12 ***Definition*** (Lowen 1978)

A property P_f of a fts is said to be a good extension of the property P in classical topology iff whenever the fts is topologically gnerated, say by (X,τ), then $(X,\omega(\tau))$ has property P_f iff (x,τ) has property P.

The following results obtained in Fora (1987) show that we have succeeded in defining good extensions of separation axioms.

2.13 *Theorem* (Fora 1987)

Let (X,τ) be a topological space. Then the following statements are equivalent:

(i) (X,τ) is a T_i-space,

(ii) $(x,\omega(\tau))$ is a T_i-space,

(iii) $(X,\omega(\tau))$ is a $T_{i\omega}$-space,

where $i \in \{0,1,2,2\frac{1}{2}\}$.

2.14 *Theorem* (Fora 1987)

Let (X,τ) be a topological space, $p \in$ {regular, normal} and $q \in$ {completely regular, f.h}. Then we have

(i) (X,τ) is a p spae iff $(X,\omega(\tau))$ is a ωp space.

(ii) (X,τ) is a q spae iff $(X,\omega(\tau))$ is a q space.

3. Fuzzy Product Spaces

Let us present our first result in this section.

3.1 *Theorem*

Let $\{(X_i,\tau_i) : i \in I\}$ be a collection of nonempty topological spaces. Let $\Pi\, \tau_i$ denote the tychonoff product topology on ΠX_i. Then we have $\omega(\Pi\tau_i) = \Pi\omega(\tau_i)$.

Proof.

To prove $\Pi\omega(\tau_i) \subseteq \omega(\Pi\tau_i)$, let λ be a basic open set in $\Pi\omega(\tau_i)$. Then there exist $i_1, i_2, \ldots, i_n \in I$ and there exists $\lambda_i \in \omega(\tau_i)$, $i \in I$, such that $\lambda = \Pi\lambda_i$, where $\lambda_i = 1_i$ for all $i \in I - \{i_1, i_2, \ldots, i_n\}$. For each $t \in [0,1]$, it can be easily proved that $(\Pi\lambda_i)^{-1}(t,1] = \Pi U_i$, where $U_i = X_i$ for all $i \in I - \{i_1, i_2, \ldots, i_n\}$ and $U_j = \lambda_j^{-1}(t,1]$ for all $j \in \{i_1, i_2, \ldots, i_n\}$. Since $\lambda_j \in \omega(\tau_j)$ for all $j \in I$, therefore $U_j \in \tau_j$ for $j \in I$. Moreover, one can easily observe that $\Pi\, U_i \in \Pi\tau_i$. Hence $\Pi\lambda_i \in \omega(\Pi\tau_i)$.

To prove the other inclusion, *i.e.* $\omega(\Pi\tau_i) \subseteq \Pi\omega(\tau_i)$, let $\lambda \in \omega(\Pi\tau_i)$. Take p any fuzzy point in ΠX_i such that $p \in \lambda$. Let $x = (x_i)_{i\in I}$ be the support of p. Then $p(x) < \lambda(x)$. Let $r = \frac{1}{2}(p(x) + \lambda(x))$. Then $p(x) < r < \lambda(x)$, *i.e.* $x \in \lambda^{-1}(r,1]$ and $\lambda^{-1}(r,1] \in \Pi\tau_i$. Hence there exist $i_1, i_2, \ldots, i_n \in I$ and there exists $U_i \in \tau_i$, $i \in I$, such that $x \in \Pi\, U_i \subseteq \lambda^{-1}(r,1]$ and $U_i = X_i$ for all $i \in I - \{i_1, i_2, \ldots, i_n\}$. Define $\lambda_i = r \cdot \chi_{U_i}$ for each $i \in \{i_1, i_2, \ldots, i_n\}$ and $\lambda_j = 1_j$ for all $j \in I - \{i_1, i_2, \ldots, i_n\}$. Then $\Pi\lambda_i$ is a basic open set in $\Pi\omega(\tau_i)$ and moreover we have $p \in \Pi\lambda_i \subseteq \lambda$. Hence, according to Theorem 3.2 of Wong (1974), $\lambda \in \Pi\omega(\tau_i)$.

3.2 *Theorem*

Let (X_i, Γ_i) be a fuzzy nonempty space for each $i \in I$. Then $(\Pi\, X_i, \Pi\, \Gamma_i)$ is a $T_{0\omega}$-space if and only if each (X_i, Γ_i), $i \in I$, is a $T_{0\omega}$-space.

Proof

($\leftarrow$) Let p,q be two distinct fuzzy points in ΠX_i with supports $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ respectively. Since $x \neq y$, there exists $j \in I$ such that $x_j \neq y_j$. Take p_0, q_0 to be fuzzy points in X_j such that $p_0(x_j) = p(x)$ and $q_0(y_j) = q(y)$. Then p_0, q_0 are two distinct fuzzy points in the $T_{0\omega}$-space (X_j, Γ_j). Thus there exists $\lambda_j \in \Gamma_j$ such that $p_0 \in \lambda_j \subseteq q_0'$ or $q_0 \in \lambda_j \subseteq p_0'$. In the case $p_0 \in \lambda_j \subseteq q_0'$, we get $p_0(x_j) < \lambda_j(x_j)$; so $p(x) < \lambda_j(x_j) = (\Pi\lambda_i)(x)$, where $\lambda_i = 1_i$ for all $i \in I\text{-}\{j\}$. Hence $p \in \Pi\lambda_i \subseteq q'$. In the second case, *i.e.* $q_0 \in \lambda_j \subseteq p_0'$, we get a similar situation.

($\rightarrow$) Let $(\Pi X_i, \Pi\Gamma_i)$ be a $T_{0\omega}$-space and let $j \in I$. To prove (Xj, Γ_j) a $T_{0\omega}$-space, let p_0, q_0 be two distinct fuzzy points in X_j with supports x_{jp}, and x_{jq} respectively. Since $X_i \neq \phi$, there exists $x_i \in X_i$ for each $i \in I\text{-}\{j\}$. Define $x_{ip} = x_{iq} = x_i$ for each $i \in I\text{-}\{j\}$ and put $x_p = (x_{ip})_{i\in I}$ and $x_q = (x_{iq})_{i\in I}$. Let p, q be two fuzzy points in $\Pi\, X_i$ such that $(p(x_p) = p_0(x_{jp})$ and $q(x_q) = q_0(x_{jq})$. Then p, q are distinct fuzzy points in the $T_{0\omega}$-space $(\Pi X_i, \Pi\, \Gamma_i)$. Thus; using Theorem 3.2 of Wong (1974), there exists a basic open set λ in $(\Pi\, X_i, \Pi\Gamma_i)$ such that $p \in \lambda \subseteq q'$ or $q \in \lambda \subseteq p'$. Hence there exist $i_1, i_2, \ldots, i_n \in I$ and there exists $\lambda_i \in \Gamma_i$, $i \in I$, such that $\lambda_i = 1_i$ for all $i \in I\text{-}\{i_1, i_2, \ldots, i_n\}$ and $\lambda = \Pi\lambda_i$. It is easy to observe that $p_0 \in \lambda_j \subseteq q_0'$ or $q_0 \in \lambda_j \subseteq p_0'$.

Actually, the above technique can be used to prove the following result.

3.3 *Theorem*

Let $\{(X_i, \Gamma_i) : i \in I\}$ be a collection of nonempty fuzzy spaces. Then

(i) $(\Pi X_i, \Pi\Gamma_i)$ is a T_j-space if and only if each (X_i, Γ_i) is a T_j-space; where $j \in \{0,1,2,2\frac{1}{2}, 3, 0_\omega, 1\omega, 2\omega, 2\frac{1}{2}\omega, 3\omega, 3\frac{1}{2}\omega, s.c\}$.

(ii) $(\Pi X_i, \Pi\Gamma_i)$ is regular (ω-regular) if and only if each (X_i, Γ_i), $i \in I$, is regular (ω- regular).

The following result shows that even the property of being completely regular is still fuzzy productive.

3.4 *Theorem*

Let $\{(X_i, \Gamma_i) : i \in I\}$ be a collection of nonempty fuzzy spaces. Then $(\Pi X_i, \Pi\Gamma_i)$ is a completely regular space if and only if each (X_i, Γ_i) $i \in I$, is a completely regular space.

Proof

($\leftarrow$) Let p be a fuzzy point in ΠX_i with support $t = (t_i)_{i \in I}$ and let μ be a fuzzy closed set in $(\Pi X_i, \Pi\Gamma_i)$ such that $p \in \mu'$, *i.e.* $p(t) < \mu'(t)$. Let $r = \frac{1}{2}(p(t) + \mu'(t))$. Then $p(t) < r < \mu'(t)$. Since $p \in \mu'$ and $\mu' \in \Pi\Gamma_i$, therefore; by Theorem 3.2 of Wong (1974), there exist $i_1, i_2, \ldots, i_n \in I$ and there exist $\mu_i \in \Gamma_i$ for all $i \in I$ such that $p \in \Pi\mu_i \subseteq \mu'$, where $\mu_i = 1_i$ for each $i \in I\text{-}\{i_1, i_2, \ldots i_n\}$. Let p_i be a fuzzy point in X_i with $p_i(t_i) = p(t)$. Since $p \in \Pi\,\mu_i$, therefore $p_j(t_j) = p(t) < \min\{\mu_i(t_i) : i = i_1, i_2, \ldots, i_n\} \leq \mu_j(t_j)$ for each $j \in \{i_1, i_2, \ldots, i_n\}$. Hence $p_j \in \mu_j$ for all $j \in \{i_1, i_2, \ldots, i_n\}$. Since (X_j, Γ_j) is a completely regular space, there exists a fuzzy continuous function $f_j:(X_j, \Gamma_j) \rightarrow [0,1](L)$ such that $p_j(x_j) \leq f_j(x_j)(1-) \leq f_j(x_j)(0+) \leq \mu_j(x_j)$ for all $x_j \in X_j$, where $j \in \{i_1, i_2, \ldots, i_n\}$. Now, define $f:(\Pi X_i, \Pi\Gamma_i) \rightarrow [0,1](L)$ by $f((x_i)_{i \in I}) = \min\{f_j(x_j) : j \in \{i_1, i_2, \ldots, i_n\}\}$. Notice that f is a well defined fuzzy continuous function satisfying the condition that for every $x = (x_i)_{i \in I} \in \Pi X_i$ we have $p(x) \leq f(x)(1-) \leq f(x)(0+) \leq \mu'(x)$.

($\rightarrow$) Let $(\Pi X_i, \Pi\,\Gamma_i)$ be a completely regular space and let $j \in I$. To prove (X_j, Γ_j) a completely regular space, let p_0 be a fuzzy point in X_j with support t_j. Let $\lambda_j \in \Gamma_j$ be such that $p_0 \in \lambda_j$. Since $X_i \neq \phi$, there exists $t_i \in X_i$ for each $i \in I\text{-}\{j\}$. Take p to be the fuzzy point in ΠX_i such that $p(t) = p_0(t_j)$, where $t = (t_i)_{i \in I}$. Let $\lambda = \Pi\lambda_i$ where $\lambda_i = 1_i$ for all $i \in I\text{-}\{j\}$. Then $\lambda \in \Pi\,\Gamma_i$ and $p \in \lambda$. Since $(\Pi X_i, \Pi\Gamma_i)$ is a completely regular space, therefore there exists a fuzzy continuous function $f:(\Pi X_i, \Pi\,\Gamma_i) \rightarrow [0,1](L)$ such that for every $x = (x_i)_{i \in I} \in \Pi X_i$, we have $p(x) \leq f(x)(1-) \leq f(x)(0+) \leq \lambda(x)$. Now, define $g:(X_j, \Gamma_j) \rightarrow [0,1](L)$ by $g(x_j) = f((x_i)_{i \in I})$, where $x_i = t_i$ for $i \in I\text{-}\{j\}$. Then g is a fuzzy continuous function satisfying the condition that for any $x_j \in X_j$ we have $p_0(x_j) \leq g(x_j)(1-) \leq g(x_j)(0+) \leq \lambda_j(x_j)$.

The above theorem suggests the following result.

3.5 *Theorem*

Let $\{(X_i, \Gamma_i) : i \in I\}$ be a collection of nonempty fuzzy spaces. Then $(\Pi X_i, \Pi\Gamma_i)$ is functionally Hausdroff if and only if each (X_i,Γ_i) is functionally Hausdorff, $i \in I$.

Observe that the theorem is not true for the ω-normal spaces. In fact, we have only one direction is true. To illustrate this fact, we have the following result.

3.6 *Theorem*

(a) Let $\{(X_i, \Gamma_i) : i \in I\}$ be a collection of nonempty fuzzy spaces. If $(\Pi X_i, \Pi\Gamma_i)$ is ω-normal ($T_{4\omega}$), then each (X_i,Γ_i), $i \in I$, is ω-normal ($T_{4\omega}$).

(b) There exists an ω-normal ($T_{4\omega}$) space (X,Γ) for which $(X \times X, \Gamma \times \Gamma)$ is not ω-normal (not $T_{4\omega}$).

Proof

(b) Let S denote the Soprgenfrey Line. Then ω(S) is ω-normal ($T_{4\omega}$) according to Fora (1987). Using Theorem 3.2, we have ω(S)xω(S) = ω(SxS) which is not ω-normal (not $T_{4\omega}$) according to Fora (1987).

It is a remarkable notice that if $X_1 = X_2 = \{x\}$, $\tau_1 = \{0, r : \frac{1}{2} \leq r \leq 1\}$ and $\tau_2 = \{1, r : 0 \leq r \leq \frac{1}{2}\}$, then $X_1 \times X_2 = \{(x,x)\}$ and $\tau_1 \times \tau_2 = \{r : 0 \leq r \leq 1\}$. This means that the product f.t.s. $X_1 \times X_2$ is indeed a $T_{i\omega}$-space for all $i \in \{0,1,2,2\frac{1}{2}, 3,3\frac{1}{2},4\}$. Notice that the q.f.t.s (X_1,τ_1) is not even a T_s-space. For this reason we observe that constant functions are important in defining fuzzy topological spaces for the purpose of separation axioms. For more details we refer the reader to consult Ming and Ming (1980b), and Lowen and Wuyts (1988).

4. Fuzzy Connectedness and The Fixed Point Property

We start this section with the following definition.

4.1 ***Definition***

A (fuzzy) topological space (X,Γ) is said to have the fixed point property (f.p.p for short) iff every (fuzzy) continuous function $f:(X,\Gamma) \rightarrow (X,\Gamma)$ has a fixed point (*i.e.* a point $x \in X$ for which $f(x) = x$).

Using Theorem 1.6 and Theorem 1.7, we get the following result.

4.2 *Theorem*

Let (X,τ) be a topological space. Then we have

(i) (X,τ) has the f.p.p. if and only if $(X,\omega(\tau))$ has the f.p.p.

(ii) (X,τ) has the f.p.p. if and only if $(X,X|\tau)$ has the f.p.p.

In usual topological spaces, it is an easy exercise showing that a space having the f.p.p. must be connected and a T_0-space. In fuzzy topological spaces, we shall give an example of a fuzzy space have the f.p.p. and yet it is neither connected nor a $T_{0\omega}$-space.

4.3 ***Definition***

A fts (X,Γ) is called connected iff it has no clopen sets other than the constant fuzzy sets.

4.4 *Example*

There exists a fuzzy space (X,Γ) which has the f.p.p. and yet it is neither connected nor a $T_{0\omega}$-space.

Proof

Let $X = \{0,1\}$, $\Gamma = \{\lambda, \lambda', r{:}0 \leq r \leq 1\}$ where $\lambda(0) = \frac{1}{2}$ and $\lambda(1) = \frac{1}{4}$. To prove (X,Γ) has the f.p.p., we must show that the function $f{:}X \to X$; given by $f(0) = 1$ and $f(1) = 0$; is not fuzzy continuous. Indeed, $f^{-1}(\lambda)(0) = \lambda(f(0)) = \lambda(1) = \frac{1}{4}$.

Hence $f^{-1}(\lambda) \notin \Gamma$. This shows that (X,Γ) has indeed the f.p.p. It is clear that (X,Γ) is not connected because λ is a clopen set in X. Moreover, for the fuzzy points p, q in X; given by $p(0) = 0.9$ and $q(1) = 0.9$; there does not exist $\mu \in \Gamma$ such that $p \in \mu \subseteq q'$ or $q \in \mu \subseteq p'$. Hence (X,Γ) is not a $T_{0\omega}$-space.

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(Rceived 25/10/1987;
in revised form 22/05/1990)

مسلمات الفصل، الفضاءات الجزئية وجداء الفضاءات فـــي الفضاء التبولوجي السائب

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قسم الرياضيات ـ جامعة البحرين ـ ص . ب : ٣٢٠٣٨ ـ مدينة عيسى ـ البحرين

لقد عرفنا في هذا البحث مسلمات فصل متعددة على الفضاءات السائبة منها: ت$_{٠}$، ت$_{١}$، ت$_{١و}$، ت$_{٢}$، ت$_{٢و}$، ت$_{س}$، ت$_{جـ}$، ت$_{٢\frac{١}{٢}}$، ت$_{٢\frac{١}{٢}و}$، ت$_{٣}$، ت$_{٣و}$، ت$_{٣\frac{١}{٢}و}$، ت$_{٤}$، ت$_{٤و}$، كما وعرفنا الفضاءات السائبة التالية: المنتظمة، المنتظمة الضعيفة، المنتظمة التامة، الهاوسدورف الدالي، السوية والسوية الضعيفة، المترابطة وكذلك الفضاءات السائبة التي تملك خاصية النقطة الثابتة.

اثبتنا في هذا البحث مايلي :ـ

١ ـ جميع مسلمات الفصل ت$_{ن}$ المذكورة اعلاه هي خواص توبولوجية (بغض النظر عن قيمة ن) وكذلك الخواص التالية: المنتظمة، المنتظمة الضعيفة، المنتظمة التامة، الهاوسدورف الدالي، السوية، السوية الضعيفة.

٢ ـ جميع الخواص المذكورة في البند رقم ـ ١ ـ أعلاه هي جدائية ووراثية الا الخواص التالية: ت$_{٤}$، ت$_{٤و}$، السوية، السوية الضعيفة.

٣ ـ اذا كان الفضاء الجدائي π س$_{ي}$ يملك أحد الخواص المذكورة في البند رقم ـ ١ ـ أعلاه فإن جميع الفضاءات الاصلية س$_{ي}$ تملك نفس الخاصية.

٤ ـ وجود فضاء سائب يملك خاصية النقطة الثابتة ولكنه مع ذلك غير مترابط.

٥ ـ ان معظم مسلمات الفصل المعرفة في هذا البحث هي تقييم جيد لمسلمات الفصل العادية التي تملكها الفضاءات التبولوجية العادية.

لنفرض أن (س ، ت) فضاءاً توبولوجيا ولنفرض أن د(ت) تدل على مجموعة جميع الدوال شبه المستمرة السفلية المعرفة في الفضاء التبولوجي س إلى الفترة المغلقة [صفر، ١]. اثبتنا في هذا البحث أيضاً النتائج التالية :

١ ـ إذا كانت { (س$_{ن}$، ت$_{ن}$) ن ∈ ك } عائلة من الفضاءات التبولوجية فإن :

$$د\left(\prod_{ن \in ك} ت_{ن}\right) = \prod_{ن \in ك} د(ت_{ن})$$

٢ ـ يملك الفضاء التبولوجي العادي (س ، ت) خاصية النقطة الثابتة اذا وفقط اذا ملكها الفضاء السائب (س، د(ت)) .

كما وتمت مقارنة بعض مسلمات الفصل المذكورة في هذا البحث مع مسلمات الفصل التي تم تعريفها من قبل باحثين مختلفين في هذا المجال واوضحنا اهمية وجود الاقترانات الثابتة في الفضاءات التبولوجية السائبة.