Arab Gulf J. Scient. Res., **8** (3), pp. 17-28 (1990)

The Group of Automorphisms of Finite Chain Rings

Y. Al-Khamees

Department of Mathematics, King Saud University
P.O. Box 2455, Riyadh 11451, Saudi Arabia

ABSTRACT. A chain ring is an associative ring with identity whose ideals form a chain. We determine (in certain cases) the group of automorphisms of finite chain rings.

All rings considered in this paper are associative and have an identity. Chain rings have been examined by a number of researchers. In particular Krull (1924) examined commutative chain rings and discovered the coefficient subring of a finite commmutative ring and Snapper (1952) Sharpened Krull's results. Wirt (1972) determined the structure of a finite chain ring as the quotient of a skew polynomial ring (Ore polynomial ring) over a Galois ring by an ideal of special form, generated by Eisenstein polynmial. Nechaev (1973) rediscovered almost the same as Wirt structure of finite chain rings and he called them Galois-Eisenstein-Ore rings because of the constructions involved. Fisher (1976) gave the structure of a finite chain ring as the quotient of a skew power series ring over a certain complete discrete commutative valuation domain by an ideal of special form similar to the ideal involved in the construction given by Wirt. Arkhipov (1972) classified certain commutative chain rings in to three classes and Rybkin (1981) generalized Arkhipov results to the finite chain rings. It is easy to see that a finite ring R is a chain ring if and only if the set J of all its zero divisors forms a principal ideal. In such a case, J is the unique maximal ideal of R, $|R| = p^{mr}$, $|J| = p^{(m-1)r}$, where the characteristic of R is p^n, p being a prime, m is the index of nilpotency of J (*i.e.* $J^m = .\ \{0\}$ and $J^{m-1} \neq \{0\}$), $1 \leq n \leq m$ and p^r is the residue order of R (*i.e.* R/J is a field of order p^r) (*cf.* Raghavendran 1969).

Since J is nilpotent it makes sense to consider the greatest positive integer k such that p is an element of J^k. It turns out that this integer plays an important role in the case of finite chain rings. Clark and Liang (1973) have determined the number of isomorphism classes of finite commutative chain rings with (k, p) = 1. This enumeration is generalized by the author for the finite chain rings which are not necessarily commutative (Al-Khamees 1981).

In(Al-Khamees 1989a), the group of automorphisms of finite chain rings of characteristic p is determined. In this paper, we determine the group of automorphisms for the finite commutative chain rings with (k,p) = 1 and then show how this can be used to settle the non-commutative case with (k,p) = 1. As a matter of fact we determine the group of automorphisms of finite chain rings in terms of the group of automorphisms of certain finite commutative chain rings under weaker condition that (k,p) = 1 (see remark 4). It is perhaps worth noting that finite principal ideal rings are direct sums of finite chain rings.

Let R be a finite chain ring and p, n, m, r be integers associated to it as described above. If n = m, then $R = Z_{p^n}[a]$, where Z_{p^n} is the ring of the integers modulo p^n and a an element of R of multiplicative order p^r-1. In this case Aut R, the automorphism group of R, is cyclic and is of order r. These rings are uniquely determined by the triplet, p. n, r; denoted by GR (p^n,r) and are called Galois rings (*cf.* Raghavendran 1969).

Let R be a finite chain ring and p, n, r, k, m be as described above. It is already known (Clark 1972) that R has a coefficient subring S (*i.e.* R/J(R) is isomorphic to S/pS) of the form $GR(p^n,r)$; moreover any two coefficient subrings of R are conjugate in R. Let $J = R\pi$. Then we have the following facts due to Clark and Drake (1973) and Wirt (1972). There exists a polynomial

$$f = x^k - pu\,(1 + \sum_{i=1}^{k-1} r_i x^i)$$

such that π is a root of f, where u is an element of $< a >$ and r_i are elements of S. In fact, we can choose π in such a way that there exists an automorphism σ of S such that

$$R = \sum_{i=0}^{k-1}{}^{\oplus} S\pi^i \quad (\text{as } S - \text{modules})$$

and for each element r of S, $\pi r = r^\sigma \pi$. It is easy to see that R is isomorphic to $S[x, \sigma]/(f, p^{n-1}x^t)$, where $S[x, \sigma]$ is the skew polynomial ring with respect to σ, $1 \leq t \leq k$ and $t = m - (n-1)k$. It is known that if $n > 1$ then $\sigma^k = id_{R_0}$ and that σ is

uniquely determined by R and S (Al-Khamees 1981). Therefore we call σ the associated automorphism of R with respect to S. Throughout this paper for a given finite chain ring R, we denote by T_R the set of all triplets (S, σ, π) which come from the above description. Let k' be the order of σ. We observe that isomorphic finite chain rings have the same set of integers p, n, r, k, k', m associated to them, therefore we call these integers, p, n, r, k, k', m invariants of the finite chain ring R. However it is to be noted that these invariants do not determine a finite chain ring completely. Also, in the case of a finite commutative chain ring R the integer $k' = 1$, and therefore we have only five integers p, n, r, k, m as invariants of R.

Let R be a finite chain ring with invariants p, n, r, k, k', m with $n > 1$ and R_1 be the centralizer of S in R. Then R_1 is a commutative chain subring of R, $J(R_1) = R_1\pi^{k'}$ and

$$R_1 = \sum_{i=0}^{k_1-1} \oplus S\pi^{k'i},$$

where $J(R_1)$ is the Jacobson radical of R_1 and $k_1 = k/k'$. The integers p, n, r, k_1, m_1, where $m_1 = [(m/k') + 1]$, are invariants of the commutative chain subring R_1 in the sense explained above (Al-Khamees 1981). Clearly, if Z(R) is the centre of R, then

$$Z(R) = \sum_{i=0}^{k_1-1} \oplus S_o\pi^{k'i} + \Omega,$$

where $S_o = Z_{p^n}[b]$ is a Galois subring of S of the form $GR(p^n, r')$, $r' = r/k'$, $b = a^e$, $e = p^r - 1/p^{r'} - 1$ and Ω is either $J(R)^{m-1}$ or zero according as k' divides $m-1$ or otherwise. As π^k is an element of Z(R), if we write $\pi^k = pux$, then we get that, for $n > 2$ or $t > 1$, u is an element of $< b >$ otherwise u is an element of $< a >$. Also, if $(k, p) = 1$ then we can choose π in such a way that $\pi^k = pu$, that is, in this case (Al-Khamees 1981) we can choose an associated (very pure) Eisenstein polynomial of R over S of the form $x^k - pu$. It is easy to establish that if a chain ring is very pure over a particular coefficient subring then it is very pure over any other coefficient subring. From the construction of the finite chain ring R given before as the quotient of skew polynomial ring over its coefficient subring S with respect to the associated automorphism σ, one can deduce easily that the integers p, n, r, k, m with triplet σ, u, x determine completely the finite chain ring R. Therefore we call the set of entities p, n, r, k, m, σ, u, x the invariants of the finite chain ring R. In the case of a commutative finite chain ring R these eight invariants reduce to seven, as we observe that the associated automorphism σ of R with respect to its coefficient subring S is the identity automorphism. Also, in the case $(k, p) = 1$, the invariants of the finite chain ring R reduce to six or seven according to whether the ring is commutative or not.

Unless otherwise stated all symbols introduced above will retain their meanings throughout the paper. In addition, let R_2 be the finite chain ring with the invariants p,n', r, k, m', σ, u, x with $n' > 2$, where $m' = (n'-1)k+t$ and let $R' = R_2/(J(R_2))^{m'-k}$; we say that the ring R' is derived from the ring R. Also suppose that Aut $S = < \tau >$, $H_i(R) = 1+J^i(R)$, $H_j'(R) = H_i(R)/H_{m-1}(R)$, $F = S_o/pS_o$, $K = S/pS$, $k_2 = lk_1$, $l = p^{fk'}-1/p^f-1$, $a^\sigma = a^{p^f}$, $Aut_S R$ is the subgroup of Aut R fixing S elementwise. If

$$\omega = 1 + \sum_{i=1}^{k_1-1} r_i \pi^{k'i}$$

is an element of $H_1(R_1)$, we define its norm $N_\sigma(\omega)$ by

$$N_\sigma(\omega) = \alpha_o(\omega) \ldots \alpha_{\sigma^{k'-1}}(\omega),$$

where

$$\alpha_\sigma(\omega) = 1 + \sum_{i=1}^{k_1-1} r_i^\sigma \pi^{k'i}.$$

Lemma 1

Let R be a finite chain ring. Then (R_o, θ, σ) is an element of T_R if and only if $\theta = \lambda\omega\pi$, where λ is an element of $< a >$ and ω an element of $H_1(R_1)$.

The proof uses a technique similar to the one in the proof of proposition (3)(i) in (Al-Khamees 1981) and thus it is omitted here.

Lemma 2

Let R be a finite chain ring with invariants p, n, r, k, k', m with (k', p) = 1 and let $M = \{\alpha_\sigma(\omega)\ \omega^{-1} : \omega \text{ is an element of } H_1(R_1)\}$. Then

$$H_1(R_1) = M \times H_1(Z(R)).$$

Proof

Consider the homomorphism ϕ from $H_1(R_1)$ to $H_1(Z(R))$ defined by $\phi(\omega) = N_\sigma(\omega)$. Then from lemma 1 in(Al-Khamees1989b), we have $M = \text{Ker } \phi$ and $\text{Im } \phi = H_1(Z(R))$. Also $\text{Ker } \phi \cap \text{Im } \phi = \{\omega:\omega \text{ is an element of } H_1(Z(R)) \text{ and } \omega^{k'}=1\}$, which is clearly equal to the trivial subgroup. Thus $H_1(R_1) = M \times H_1(Z(R))$.

Remark 1

Let R be a finite chain ring with invariants p, n, r, k, k', m with $n > 1$ and let c be the least positive integer less than or equal to r such that

$$u^{(p^c-1)/k_2}$$

is an element of $< a >$. Put

$$u_i = u^{(p^{ic}-1)/k_2},$$

where $1 \leqslant i \leqslant s$ and $s = r/(r,c)$, and define the mapping χ from R to R as follows:

$$\chi\ (\sum\ r_j\pi^j) = (\sum\ r_j^{\rho}\ (u_1\pi)^j),$$

where $\rho = \tau^c$. Then one can easily check that χ is an automorphism of R, $\chi^i(\pi) = u_i\pi$ and the restriction of $< \chi >$ to S is $< \tau^c >$. We shall see later that the elements of $< \tau^c >$ are the only automorphisms of S which can be extended to automorphisms of R.

Proposition 1

Let R be a finite chain ring with the invariants p, n, r, k, m, σ, u with $(k,p) = 1$ and $n > 1$. Then ϕ is an automorphism of R if and only if ϕ is determined by

$$\phi\ (\sum\ r_j\pi^j) = y\ (\sum\ r_j^{\rho}\ (\lambda u_i\omega\pi)^j)\ y^{-1},$$

where y is an element of $H_1(R)$, $\rho = \tau^{ic}$, $1 \leqslant i \leqslant s$, λ an element of the subgroup A of $< a >$ of order (k_2, p^r-1) and ω an element of $H_{m-k}\ (Z(R))$.

Proof

Let ϕ be an automorphism of R, then $\phi(S)$ is a coefficient subring of R. Hence there exists a unit y_1 in R such that $\phi(S) = y_1Sy_1^{-1}$. Let ψ be the composition of the conjugation by y_1^{-1} and ϕ; then ψ restricted to S is an automorphism ρ of S and $(S, \psi(\pi),\sigma)$ an element of T_R. Thus by lemma 1, $\psi(\pi) = \mu\zeta\pi$, where μ is an element of $< a >$ and ζ an element of $H_1(R_1)$. Thus

$$pu^{\rho} = \phi(\pi^k) = (\phi(\pi))^k = (\mu\zeta\pi)^k = p(N_\sigma(\mu))^{k_1}(N_\sigma(\zeta))^{k_1}\,u\,. \qquad \ldots\ (1)$$

We can assume that $y_1 = \beta y_2$, $\mu = \lambda_1\mu_1$ and $\zeta = \zeta_1\omega$, where β is an element of $< a >$, y_2 an element of $H_1(R)$, λ_1 an element of A, μ_1 an element of $< a >$, ζ_1 an element of M and ω an element of $H_1(Z(R))$. Suppose $\lambda = \alpha_\sigma(\beta^{-1})\beta\lambda_1$, $\zeta_1 = y_3\alpha(y_3^{-1})$ and $y = y_2y_3$, where y_3 is an element of $H_1(R_1)$. From equality (1), we deduce that

$$(N_\sigma(\mu_1))^{k1} = u^\rho u^{-1} \text{ which implies that } \mu_1^{k_1k_2} = u^\rho u^{-1}.$$

Thus $\rho = \tau^{ic}$ and $\mu_1 = u_i$ for some $1 \leqslant i \leqslant s$. Also from equality (1), we have $p(N_\sigma(\omega))^k = p$. However ω being an element of $H_1(Z(R))$, we have $p\omega^k = p$ and consequently ω is an element of $H_{m-k}(Z(R))$ as $(k,p) = 1$.

Notation

Let R be the finite chain ring with the invariants p, n, r, k, m, σ, u, x with $n > 1$ and R' be the finite chain ring derived from R. Also assume A be the subgroup of $< a >$ of order (k_2, p^r-1) and s be as defined in remark 1. Put $T = R_1[x]/(x^1-\alpha)$, where $\alpha = \pi^{k'}$ and k' is the order of σ. It is easy to see that T is the commutative chain ring with the invariants p, n, r, k_2, m_2, u, x where $m_2 = lm_1$. Suppose T' is the finite chain ring derived from T. For ω an element of $H_1(R_1)$, ϕ an automorphism of R and Ω an ideal of R, let $\omega' = \omega(H_{m-1}(R))$ and ϕ' be the induced automorphism of R/Ω. Also for a unit y in R and ζ an element of $H_1(R_1)$, let ψ_y denote the conjugation by y and ϕ_ζ be the mapping from R to R defined as follows:

$$\phi_\zeta(\sum r_j\pi^j) = \sum r_j(\zeta\pi)^j.$$

Let N be the subgroup of Inn(R) which contains all the automorphisms ψ_μ, where μ is an element of $< a >$. Obviously $N = < \psi_a >$ and hence it is isomorphic to

$$< a^{p^{r-1}} >$$

Proposition 2

Let R be the finite commutative chain ring with the invariants, p, n, r, k, m, u with $(k,p) = 1$ and $n > 2$. Then

$$\text{Aut } R' \cong (A \times_\theta < \chi >) / < \chi^s >$$

where if λ an element of A, then $\theta(\chi)(\lambda) = \chi(\lambda)$.

Proof

Clearly R' is a commutative chain ring with the invariants p, n−1, r, m−k, u. Let S' be the coefficient subring of R'. From the last proposition an automorphism ϕ of R' is determined by

$$\phi\ (\sum\ r_j\pi^j)\ =\ \sum\ r_j^{\rho}\ (\lambda\ u_i\ \pi)^j,$$

where $\rho = \tau^{ic}$, $1 \leqslant i \leqslant s$, and λ is an element of A. Obviously $\mathrm{Aut}_{S'}$ R' is the subgroup of Aut R' which contains all the automorphisms ϕ_λ, where λ is an element of A. It is easy to check that χ^S is an element of Aut_S R'. Also if ϕ is an element of $<\chi>$ and of $\mathrm{Aut}_{S'}$ R', then ϕ fixing S' elementwise. But the only elements of $<\chi>$ fixing S' elementwise are in $<\chi^s>$. Therefore Aut_S R' $\cap <\chi> = <\chi^s>$. Let us form the semidirect product $\mathrm{Aut}_{S'}$ R' $\times_\theta <\chi>$ with $\theta(\chi)(\phi_\lambda) = \phi_{\chi(\lambda)}$. It is easy to verify that the correspondence f from $\mathrm{Aut}_{S'}$ R' $\times_\theta <\chi>$ to Aut R' determined by $f(\phi_\lambda,\chi) = \phi_\lambda\chi$, is a surjective homomorphism and Ker f is isomorphic to $<\chi^s>$. Now if we identify $\mathrm{Aut}_{S'}$ R' with A and Ker f with $<\chi^s>$, we get the required result.

Remark 2

Let e_{ij} denote the matrix with the identity of K in the (i,j)−position and zeros elsewhere. The group $G_{k+t-1}(K)$ of all 1-triangular matrices

$$I\ +\ \sum_{j<i}\ b_{ij}\ e_{ij}\ ,$$

where b_{ij} are elements of K, is a sylow p-subgroup of $GL_{k+t-1}(K)$ (*cf.* Weir 1955). Let E(k,t;K) be the subgroup of $G_{k+t-1}(K)$ which contains all the matrics $[a_{ij}]$, where a_{ij} is zero for all $2 \leqslant i \leqslant t$, and

$$a_{ij}\ =\ \sum_{e+f=i\ \mathrm{or}\ k+i}\ a_{e,1}\ a_{f,j-1},$$

that is $[a_{ij}]$ are determined by the first column. Obviously E(k,t;K) is of order $p^{(k-1)r}$.

Theorem 1

Let R be the finite commutative chain ring with the invariants p, n, r, k, m, u with (k,p) = 1 and n > 1. Then

$$\mathrm{Aut}\ R\ \cong\ H(R)\ \times_\theta\ \mathrm{Aut}\ R',$$

where $H(R) = H'_{m-k}(R)$ if $n > 2$ and $H(R) = E(k,t;K)$ otherwise. Also if ω' is an element $H'_{m-k}(R)$ and $[a_{ij}]$ an element of $E(k,t;K)$, then

$$\theta(\phi_\lambda\chi)(\omega') = \phi'_{\lambda\chi'}(\omega') \text{ and } \theta(\phi_\lambda\chi)([a_{ij}]) = [\lambda^{i-j}\, a_{ij}^{\rho}],$$

where ρ is the restriction of χ to R_o.

Proof

Let G be the subgroup of Aut R which contains all the automorphisms ϕ_ω, where ω is an element of $H_{m-k}(R)$. It is easy to check that Aut R is isomorphic to $G \times_\theta$ Aut R, where $\theta(\phi_\lambda\chi) = \phi_{\psi(\omega)}$ and $\psi = \phi_\lambda\chi$. It is clear that if $n > 2$, then G is isomorphic to $H'_{m-k}(R)$. Also if $n = 2$, then it is easy to see that the mapping g from G to E(k,t;K) given by $g(\phi_\omega) = [a_{ij}]$, where if

$$\omega = 1 + \sum_{i=1}^{m-1} a_{i+1,1}\,\pi^i \text{ then } a_{ij} = \sum_{e+f=i \text{ or } k+i} a_{e,1}\; a_{f,j-1}$$

is an isomorphism from G to E(k,t;K). Now by identifying G with H(R), it is easy to deduce that θ can be expressed as mentioned in the statement of the theorem.

Theorem 2

Let R be the chain ring with the invariants p, n, r, k, m, σ, u with (k,p) = 1 and $n > 1$. Then

$$\text{Aut } R \cong (\,(\text{Inn}(R)/N) \times_{\theta_1} H(Z(R))\,) \times_{\theta_2} \text{Aut } T,$$

where

$H(Z(R)) = H'_{m_1-k_1}(Z(R))$ for $n > 2$ and $H(Z(R)) = E(k_1,t_1;F)$ otherwise.

Proof

From proposition 1, an automorphism ϕ of R is determined by

$$\phi\,(\sum r_j\pi^j) = y\,(\sum r_j^{\rho}\,(\lambda\, u_i\, \omega\, \pi)^j)\; y^{-1},$$

where y is an element of $H_1(R)$, $1 \leq i \leq s$, λ is an element of A and ω an element of $H_{m-k}(Z(R))$. Let G be the subgroup of Aut R which contains all the

automorphisms $\psi_y\phi_\omega$, G_1 the subgroup of G which contains all the automorphisms ψ_y, G_2 the subgroup which contains all the automorphisms ϕ_ω, and G_3 the subgroup of Aut R which contains all the automorphisms $\phi_\lambda\chi^i$. Then it is easy to see that

$$G = G_1 \times_{\theta_1} G_2, \quad \text{Aut } R = G \times_{\theta_2} G_3$$

and G is isomorphic to Inn(R)/N. Using an argument similar to that in the proof of theorem 1, one can deduce easily that G_2 is isomorphic to H(Z(R)). Finally from proposition 1, G_1 must be equal to Aut T'.

Remark 3

It may be worth noting that θ_1 and θ_2 mentioned in the statement of the last theorem can be determined easily, and that H(Z(R)) is isomorphic to Aut Z(R)/ Aut (Z(R))', where (Z(R))' is the finite chain ring derived from Z(R).

Remark 4

Let R be a chain ring with the invariants p, n, r, k, m, σ, u, x with (k',p) = 1, where k' is the order of σ. If y is the unique element in $H_1(R_1)$ such that $y = x^{1/k'}$ and $k_3 = (k_1, p^r-1)$; then one can prove that

$$\text{Aut } R \cong (\text{Inn}(R)/N) \times_{\theta_1} (\text{Aut } Y/\text{Aut } X') \times_{\theta_2} \text{Aut } T',$$

where Y is the commutative chain ring with the invariants p, n, r', k_1, m_1, u, y (it has the same invariants as Z(R) except that y replaces x), X is the commutative chain ring with the invariants p, n, r', k_3, m_1, u and X' is the finite chain ring derived from X.

Acknowledgement

The author would like to express his gratitude to B. Corbas and S. Deshmukh for having gone through an earlier version of this paper and to the referees for their valuable remarks about this article.

References

Alkhamees, Y. (1981) The enumeration of finite principal completely primary rings, *Abhandlungen Math. Sem. Uni. Hamburg*, **51:** 226-231.

Alkhamees, Y. (1989a) The determination of the group of automorphisms of a finite chain ring of characteristic p, submitted for publication.

Alkhamees, Y. (1989b) The enumeration of finite chain rings, submitted for publication.

Arkhipov L. M. (1972) Finite rings of prinicipal ideals, *Mat. Zametki* **12**(4): 373-379.

Clark, W.E. (1972) A coefficient ring for finite non-commutative rings, *Proc. Amer. Math. Soc.* **33**(1): 25-27.

Clark, W.E. and **Drake, D.A.** (1973) Finite chain rings, *Abhandlungen Math. Sem. Uni. Hamburg* **39:** 147-153.

Clark, W.E., and **Liang, J.J.** (1973) Enumeration of finite commutative chain rings, *J. of Alegbra* **27:** 445-453.

Raghavendran, R. (1969) Finite associative rings, *Compositio Math.* **21**(2): 195-229.

Fisher J. L. (1976) Finite principal ideal rings, *Canad. Math. Bull.* **19**(3): 277-283.

Krull W. (1924) Algebraische Theorie der Ringe II, *Math. Ann.* **91:** 1-46.

Nechaev A. A. (1973) Finite rings of principal ideals, *Mat. Sb.* **91**(3): 350-366.

Rybkin A.S. (1981) Finite local rings of pricipal ideals, *Mat. Zametki* **28**(1): 465-472.

Snapper E. (1952), completely primary rings IV, *Annals of Math.* **55**(1): 46-64.

Weir, A.J. (1955) Sylow p-subgroups of the general linear group over finite fields of characteristic p, *Proc. Amer. Math. Soc.* **6:** 454-464.

Wirt, B.R. (1972) *Finite non-commutative local rings*, Ph.D. Thesis, University of Oklahoma.

(Received 14/10/1989;
in revised form 28/08/1990)

زمرة التشاكلات الذاتية لحلقات مسلسلة ومنتهية

يوسف الخميس

قسم الرياضيات ـ جامعة الملك سعود
ص. ب: ٢٤٥٥ ـ الرياض ١١٤٥١ ـ المملكة العربية السعودية

سنفرض أن الحلقات التي سنتعرض لدراستها في هذا البحث دائمة ولها عنصر محايد. تعرّف الحلقة المسلسله بأنها حلقة تشكل مثالياتها سلسلة. يمكن البرهان على أن الحلقة المنتهية تشكل حلقة مسلسلة اذا واذا فقد كانت مجموعة قواسم الصفر J فيها تشكل مثالياً رئيسياً وهو المثالي الأعظمى الوحيد للحلقة.

لما كان J مثالياً عديم القوى nilpotent (أي أن $J^m = 0$ حيث m عدد صحيح موجب) (أنظر Raghavendran 1969) فإنه من المناسب اعتبار أكبر عدد موجب p بحيث ان p عنصر من J^k، حيث p^n هو مميز الحلقة. سيتضح بعد ذلك أن العدد k يلعب دوراً مهماً في دراسة الحلقات المسلسله المنتهية.

لقد حدد كلارك ولنج (١٩٧٣) عدد الفصول المتشاكلة لحلقات مسلسلة إبداليه ومنتهية والتي يكون فيها $(k,p) = 1$ وعممت هذه النتيجة من قبل المؤلف في (١٩٨١) لحلقات مسلسله ومنتهيه وليست من الضروري إبدالية.

لقد عينت في عمل سابق (الخميس ١٩٨٩ أ) زمرة التشاكلات الذاتية لحلقات مسلسله ومنتهيه والتي يكون مميزها P. سنعين في هذا البحث زمرة التشاكلات الذاتية لحلقات مسلسلة إبدالية ومنتهية والتي يكون فيها $(k,p) = 1$.

ثم يتم توضيح كيفية إستخدام ذلك في الحالة غير الابدالية مع $(k,p) = 1$. في

الحقيقة لقد تم تعيين زمرة التشاكلات الذاتية لحلقات مسلسلة ومنتهية اعتماداً على زمرة التشاكلات الذاتيـة لحلقات مسلسلة إبـدالية منتهيـة تحت شرط أضعف من $(k,p) = 1$ (أنظر الملاحظة (4)).

قـد يكون من المنـاسب الاشارة إلى أن الحلقـات المنتهيه الـرئيسيـه هي جمـع مباشر لحلقات مسلسلة ومنتهية.